Coherence-By-Normalization for Linear Multicategories

Federico Olimpieri

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Summary

Lambek's Idea: Categories as Deductive Systems

Category Theory

Objects $A, B, C \ldots$

Morphisms $A \xrightarrow{f} B$

Identities $A \xrightarrow{\mathrm{Id}_A} A$

Composition

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] \arrow[loop left, "\mathrm{id}_A"] & B \arrow[d, "g"] \arrow[loop right, "\mathrm{id}_B"] \\ & C \arrow[loop right, "\mathrm{id}_c"'] \end{tikzcd}$$

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\end{tikzcd}$$

Proof Theory

Formulae $A, B, C \ldots$

Proofs $\dfrac{\pi}{A \vdash B}$

Axioms $\dfrac{}{A \vdash A}$

Cut rule

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}\ \text{cut}$$

Associativity and *identity* laws:
quotient proofs up to *cut-elimination*.

Two approaches to proof systems (Gentzen):

Sequent Calculus	**Natural Deduction**
$\dfrac{\Gamma \vdash A \qquad \Gamma, A \vdash B}{\Gamma \vdash B}$	Cut derivable: substitution operation
$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$ right	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$ intro
$\dfrac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}$ left	$\dfrac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}$ elim

Curry-Howard Correspondence

Natural Deduction	**λ-Calculus**
$A := o \mid A \Rightarrow B$	$A := o \mid A \Rightarrow B$
$\Gamma = A_1, \ldots, A_n$	$\Gamma = x_1 : A_1, \ldots, x_n : A_n$
$\dfrac{}{\Gamma, A, \Gamma' \vdash A}$	$\dfrac{}{\Gamma, x : A, \Gamma' \vdash x : A}$
$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$	$\dfrac{\Gamma, A \vdash M : B}{\Gamma \vdash \lambda x^A . M : A \Rightarrow B}$
$\dfrac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}$	$\dfrac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$

Proof-Theoretic Coherence

Free categories with structure can be presented by *type systems*:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & \Lambda(X) \arrow[d, dotted] \\ & A \end{tikzcd}$$

The category $\Lambda(X)$:

- Objects are given by *types*, freely generated on X.
- Morphisms are *terms* up to congruence (β and η laws).

Coherence

Two structural morphisms are equal when the corresponding terms are congruent.

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- Starting point of an *algebraic* understanding of proof systems/term calculi.
- Mathematical definition of *denotational semantics* [Lambek-Scott, 1988].
- Inspiration for Kelly and MacLane proof of coherence for closed categories [Kelly-Maclane, 1971].
- Since then, this method has been exploited to present many structures [Mints, 1981; Fiore-Saville, 2019-2021].

An Established Tradition

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What About Contexts?

Existence of a cartesian product $A \,\&\, B$ (monoidal in more general settings).
However

$$x_1 : A_1, \ldots, x_n : A_n \vdash M : B \qquad \neq \qquad x : (A_1 \,\&\, \cdots \,\&\, A_n) \vdash M : B$$

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$$
\begin{tikzcd}
A \times B \arrow[r] \arrow[dr] & A \otimes B \arrow[d, dotted] \\
 & C
\end{tikzcd}
$$

Colored Operads: Taking Contexts Seriously

Given a set X, X^* is the set of finite sequences on X.

Colored Operads (aka Multicategories)

A collection of *objects* $\mathcal{M}_0$.

A Collection of *(multi)morphisms* $\mathcal{M}(\Gamma; B)$ for $\Gamma \in \mathcal{M}_0^*, B \in \mathcal{M}_0$.

An *identity* $\mathrm{Id}_a \in \mathcal{M}(a; a)$ for $a \in \mathcal{M}_0$.

Composition:

$$- \circ (-) \;:\; \mathcal{M}(A_1, \ldots, A_n) \times \prod_{i=1}^{k} \mathcal{M}(\Gamma_i, A_i) \;\to\; \mathcal{M}(\Gamma_1, \ldots, \Gamma_n; B)$$

$$(f, (g_1, \ldots, g_n)) \;\mapsto\; f \circ (g_1, \ldots, g_n)$$

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Free Symmetric Strict Monoidal Categories

Let $[k] = \{1, \ldots, k\}$. Given a category A, we define $\mathrm{Sym}(A)$:

Free Symmetric Strict Monoidal Category on A

Objects are sequences in $\mathrm{ob}(A)^*$.

Morphisms are freely generated:

$$\frac{\sigma : [k] \to [k] \qquad f_1 : a_{\sigma(1)} \to b_1 \ \cdots \ f_k : a_{\sigma(k)} \to b_k}{\langle f_1, \ldots, f_k \rangle \sigma : \ a_1, \ldots, a_k \ \to \ b_1, \ldots, b_k}$$

Generalized Species of Structures [Fiore-Gambino-Hyland-Winskel, 2007]

Object are small categories.

1-cells $F : A \nrightarrow B$ are *generalized species*: functors $\mathrm{Sym}(A)^{op} \times B \to \mathrm{Set}$.

2-cells are natural transformations.

Identities $\mathrm{Id}_A : A \to A$ are hom-functors.

For $F : A \nrightarrow B$, $G : B \nrightarrow C$, their composition:

$$(G \circ F)(\Gamma; c) = \int^{\Delta \in \mathrm{Sym}(B)} G(\Delta; c) \times F^!(\Gamma; \Delta)$$

where

$$F^!(\Gamma; b_1, \ldots, b_n) = \int^{\Gamma_i \in \mathrm{Sym}(A)} \prod_{i=1}^{n} F(\Gamma_i; b_i) \times \mathrm{Sym}(A)(\Gamma; \Gamma_1, \ldots, \Gamma_n)$$

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Monads of Species

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A *monad* of species consists of an endospecies $M : A \nrightarrow A$, equipped with

$$\mathsf{sub} : M \circ M \to M \qquad \mathsf{id} : \mathsf{Id}_A \to M$$

satisfying the usual commutative diagrams for monads.

$$M : \mathsf{Sym}(A)^{op} \times A \to \mathrm{Set}$$

where A set.

$$- \cdot \sigma \ : \ M(a_1, \ldots, a_n; b) \ \to \ M(a_{\sigma(1)}, \ldots, a_{\sigma(n)}; b)$$

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(Symmetric) Multicategories as Monads of Species [Gambino-Joyal, 2017]

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\begin{aligned}
\mathrm{sub}_{\Gamma,b} : \quad \int^{\Delta \in \mathrm{Sym}(B)} M(\Delta; c) \times M^{!}(\Gamma; \Delta) \quad &\to \quad M(\Gamma; c) \\
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\end{aligned}
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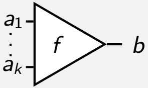
Multicategories

Objects a, b, c . . .

Morphisms



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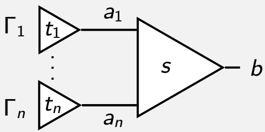

A Refined Correspondence

Multicategories

Objects a, b, c . . .

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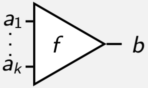

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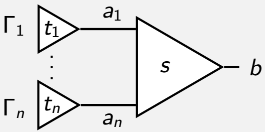

Term Calculi

Types a, b, c . . .

Terms

$$x_1 : a_1, \ldots, x_n : a_n \vdash s : b$$

Substitution

$$\frac{x_1 : a_1, \ldots, x_n : a_n \vdash s : b \qquad \Gamma_i \vdash t_i : a_i}{\Gamma_1, \ldots, \Gamma_n \vdash s\{t_1, \ldots, t_n / x_1, \ldots, x_n\} : b}$$

Substitution is an operation on terms, satisfying *associativity* and *identity* laws.

Operadic Approach to Term Calculi and Programming Languages

- Abstract syntax and variable binding [Fiore-Plotkin-Turi 1999; Tanaka-Power, 2006].
- Formal languages for structured operads [Curien-Obradović, 2017].
- Operadic presentation of untyped λ-calculus [Hyland, 2013].
- Operadic theory of program approximations [Mazza-Pellissier-Vial, 2017].
- Presentation of combinatory logic *via* abstract clones [Saville, 2024].

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Summary

A *tensor product* for $\mathcal{M}$:

For all $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \mathcal{M}_0$, a *tensor* $(a_1 \otimes \cdots \otimes a_k) \in \mathcal{M}_0$.

For all $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \mathcal{M}_0$, a *universal morphism*

$$\mathrm{re}_{a_1,\ldots,a_k} : a_1, \ldots, a_k \to (a_1 \otimes \cdots \otimes a_k)\,.$$

The map

$$\mathcal{M}(\Gamma, (a_1 \otimes \cdots \otimes a_k), \Delta; b) \xrightarrow{-\circ(\Gamma, \mathrm{re}_{a_1,\ldots,a_k}, \Delta)} \mathcal{M}(\Gamma, a_1, \ldots, a_k, \Delta; b)$$

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is *invertible*.

The Adjunction

$$\begin{tikzcd} \mathrm{Set} & \mathsf{Rep} \arrow[l, bend left] \end{tikzcd}$$

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The Representable Calculus [O., 2024]

Grammar of Types and Terms

Representable *types*:

$$\mathrm{Ty}(X) \ni a ::= o \in X \mid (a_1 \otimes \cdots \otimes a_k) \qquad k \in \mathbb{N}$$

Representable *terms*:

$$\Lambda_{\mathrm{rep}}(X) \ni s, t ::= x \mid \langle s_1, \ldots, s_k \rangle \mid s[\langle x_1^{a_1}, \ldots, x_k^{a_k} \rangle := t] \qquad (k \in \mathbb{N})$$

Examples

$$\langle x, y \rangle [\langle x, y \rangle := z]$$

$$\langle x \rangle [\langle x \rangle := (\langle z \rangle [\langle z \rangle := y])] \qquad \langle x \rangle [\langle x \rangle := \langle z \rangle][\langle z \rangle := y]$$

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Representable Type System

$$\frac{a \in \mathrm{Ty}(X)}{x : a \vdash x : a} \qquad \frac{\Gamma_1 \vdash s_1 : a_1 \quad \cdots \quad \Gamma_k \vdash s_k : a_k \quad \Gamma_i \cap \Gamma_j = \emptyset}{\Gamma_1, \ldots, \Gamma_k \vdash \langle s_1, \ldots, s_k \rangle : (a_1 \otimes \cdots \otimes a_k)}$$

$$\frac{\Gamma_1 \vdash t : (a_1 \otimes \cdots \otimes a_k) \quad \Gamma_2, x_1 : a_1, \ldots, x_n : a_k, \Gamma_3 \vdash s : b \quad \Gamma_i \cap \Gamma_j = \emptyset}{\Gamma_2, \Gamma_1, \Gamma_3 \vdash s[\langle x_1^{a_1}, \ldots, x_k^{a_k} \rangle := t] : b}$$

Term Rewriting

Substitution on Terms

$$\frac{\Delta_1, x_1 : a_1, \ldots, x_k : a_k, \Delta_2 \vdash s : b \qquad \Gamma_i \vdash t_i : a_i}{\Delta_1, \Gamma_1, \ldots, \Gamma_k, \Delta_2 \vdash s\{t_1, \ldots, t_k / x_1, \ldots x_k\} : b}$$

satisfying *associativity* and *identity* laws.

β and η

$$\Gamma \vdash s[x_1^{a_1}, \ldots, x_k^{a_k} := \langle t_1, \ldots, t_k \rangle] : b \quad \to_\beta \quad \Gamma \vdash s\{t_1, \ldots, t_k / x_1, \ldots, x_k\} : b$$

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Termination

$\rightarrow^* \subseteq \Lambda_{\text{rep}}(\Gamma; a) \times \Lambda_{\text{rep}}(\Gamma; a)$ it is the reflexive and transitive closure of $\rightarrow_\beta \cup \rightarrow_\eta$.

Substitution is multi-*linear*:

$$\langle x, y\rangle[\langle x, y\rangle = \langle q, v\rangle][\langle q, v\rangle := \langle r, t\rangle] \;\rightarrow_\beta^*\; \langle r, t\rangle$$

- The *size* of terms decreases under β, so the relation is terminating.
- We can also find a type-related measure for termination of (restricted) η.
- We get *strong normalization* (that is, termination) of the full reduction.

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Confluence

We also have *confluence*: given $s \to^* s_1$ and $s \to^* s_2$, there exist t s.t.

$$\begin{tikzcd}
& s \arrow[dl] \arrow[dr] & \\
s_1 \arrow[dr, dotted] & & s_2 \arrow[dl, dotted] \\
& t &
\end{tikzcd}$$

Which entails *uniqueness* of normal forms.

We have a function

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Structural Equivalence

We need additional equalities on terms:

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\langle t_1, \ldots, t_i[\vec{x} := \vec{s}], \ldots, t_k \rangle & \equiv & \langle t_1, \ldots, t_k \rangle[\vec{x} := \vec{s}] & \vec{x} \text{ not free in } t_j \\
s[\vec{x} := (t[\vec{y} := u])] & \equiv & s[\vec{x} := t][\vec{y} := u] & \vec{y} \text{ not free in } s \\
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However, we can always *postpone* structural equivalence (cfr. Accattoli-Kesner LSC):

Strong Bisimulation

if $s \equiv s'$ and $s \to t$ then there exists t', s.t. $s' \to t'$ and $t \equiv t'$.

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Free Representable Multicategories

Congruence on Terms

$$s =_{\beta\eta} t \quad := \quad s \to^* t \text{ or } t \to^* s.$$

$$s \sim_{\text{rep}} t \quad := \quad s =_{\beta\eta} t \text{ or } s \equiv t.$$

The Multicategory Rep(X)

Objects:

$$\text{Rep}(X)_0 \ni a ::= o \in X \mid (a_1 \otimes \cdots \otimes a_k)$$

Morphisms:

$$\text{Rep}(X)(\Gamma; a) \quad = \quad \{s \mid \Gamma \vdash s : a\}/\sim_{\text{rep}}$$

Composition is given by *substitution*.

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Representable Structure

Lemma (Substitution and Congruence)

If $s \sim_{\text{rep}} s'$ and $\vec{t} \sim_{\text{rep}} \vec{t'}$ then $s\{\vec{t}/\vec{x}\} \sim_{\text{rep}} s'\{\vec{t'}/\vec{x}\}$

The *universal* morphisms are sequences of variables:

$$x_1 : a_1, \ldots, x_k : a_k \vdash \langle x_1, \ldots, x_k \rangle : (a_1 \otimes \cdots \otimes a_k)$$

The representable structure:

$$\begin{tikzcd} \mathrm{Rep}(X)(\Gamma, (a_1 \otimes \cdots \otimes a_k), \Delta; b) \arrow[r, bend left, "-\{\langle x_1,\ldots,x_k\rangle/x\}"] \arrow[r, phantom, "\perp"] & \mathrm{Rep}(X)(\Gamma, a_1, \ldots, a_k, \Delta; b) \arrow[l, bend left, "-[x_1,\ldots,x_k:=x]"] \end{tikzcd}$$

$$s =_\eta s\{\vec{x}/x\}[\vec{x} := x] \qquad s[\vec{x} := \vec{x}] =_\beta s\{\vec{x}/\vec{x}\} = s$$

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Coherence Theorem

Theorem

Let $\gamma \vdash s : a$ and $\gamma \vdash s' : a$ be normal terms. Then $s \equiv s'$.

Proof.

The proof is a straightforward induction on the structure of normal forms. □

Exploiting *termination* and *confluence*, we then get:

Theorem

Let $[s], [s'] \in \mathrm{Rep}(X)(\gamma; a)$. *Then* $\mathrm{nf}(s) \equiv \mathrm{nf}(s')$.

From which we obtain *coherence* as corollary:

Theorem (MacLane)

Let $[s], [s'] \in \mathrm{Rep}(X)(\gamma; a)$. *Then* $[s] = [s']$.

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Summary

A Naive Symmetric Type System

$$\frac{a \in \mathrm{Ty}(X)}{x : a \vdash x : a} \qquad \frac{\Gamma_1 \vdash s_1 : a_1 \quad \cdots \quad \Gamma_k \vdash s_k : a_k \quad \Gamma_i \cap \Gamma_j = \emptyset}{\Gamma_1, \ldots, \Gamma_k \vdash \langle s_1, \ldots, s_k \rangle : (a_1 \otimes \cdots \otimes a_k)}$$

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$$\frac{x_1 : a_1, \ldots, x_n : a_n \vdash s : b \quad {\color{red}\sigma} : [n] \cong [n]}{x_{\sigma(1)} : a_{\sigma(1)}, \ldots, x_{\sigma(n)} : a_{\sigma(n)} \vdash s : b}$$

Lack of Canonicity

Let $\sigma : [2] \cong [2]$ be the swap permutation. The derivations

$$\dfrac{\dfrac{\dfrac{x : o \vdash x : o \qquad y : o \vdash y : o}{x : o, y : o \vdash \langle x, y\rangle : (o \otimes o)} \qquad z : o \vdash z : o}{x : o, y : o, z : o \vdash \langle\langle x, y\rangle, z\rangle : ((o \otimes o) \otimes o)} \quad \sigma + \mathrm{id}}{y : o, x : o, z : o \vdash \langle\langle x, y\rangle, z\rangle : ((o \otimes o) \otimes o)}$$

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correspond to the *same* typed term.

Restrict the choice of permutation to *shuffles* [Melliès-O.-Vaux, 2026]:

$$\dfrac{\dfrac{\dfrac{\dfrac{x : o \vdash x : o \qquad y : o \vdash y : o}{x : o, y : o \vdash \langle x, y \rangle : (o \otimes o)}\ \sigma}{y : o, x : o \vdash \langle x, y \rangle : (o \otimes o)} \qquad z : o \vdash z : o}{y, x : o, z : o \vdash \langle \langle x, y \rangle, z \rangle : ((o \otimes o) \otimes o)}}{y : o, x : o, z : o \vdash \langle \langle x, y \rangle, z \rangle : ((o \otimes o) \otimes o)}$$

A Shuffled System

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Free Symmetric Representable Multicategories

Congruence on Terms

$$s =_{\beta\eta} t \quad := \quad s \to^* t \text{ or } t \to^* s.$$

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Theorem

Given a normal form $\gamma \vdash s : a$, there exist a unique permutation $\sigma \in \mathrm{stab}(\gamma)$ and permutation-free term s' s.t. $s = s' \cdot \sigma$.

Hence, to any normal form s we can assign its *underling permutation* $\mathrm{perm}(s)$.

Theorem (MacLane)

Let $[s], [s'] \in \mathrm{SRep}(X)(\gamma; a)$. *If* $\mathrm{perm}(\mathrm{nf}(s)) = \mathrm{perm}(\mathrm{nf}(s'))$ *then* $s =_{\mathrm{srep}} s'$.

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Conclusions

- Correspondence: representable resource terms and representable multicategories.
- We also have it for closed structures [O., 2024].
- 2-dimensional setting: connection with the theory of standardization.
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Thank You!