Generalized species in denotational semantics

Species and semantics working group

Zeinab GALAL

March 20, 2025

Recall: species of structures

category $\mathbb{B}$ objects: finite sets, morphisms: bijections

Definition

A **species of structure** is a functor $F : \mathbb{B} \to \mathbf{Set}$:

- Given a finite set of labels $A \in \mathbb{B}$, an element $s \in F(A)$ is called an *F-structure on A*
- Given a bijection $\sigma : A \xrightarrow{\sim} B \in \mathbb{B}$, the bijection $F(\sigma) : FA \xrightarrow{\sim} FB$ is called the transport of F-structures along σ

Disclaimer: in this talk, I do not restrict to species $\mathbb{B} \to \mathbf{FinSet}$ whose set of structures is finite.

Joyal, "Une théorie combinatoire des séries formelles", 1981

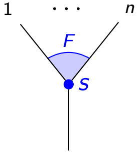
Equivalent notion: symmetric sequences

category $\mathbb{P}$ objects: natural numbers, morphisms: permutations

A **symmetric sequence** is a functor $F : \mathbb{P} \to \mathbf{Set}$ or equivalently

- a set $F(n)$ of F-structures for every $n \in \mathbb{N}$

a general element
$s \in F(n)$



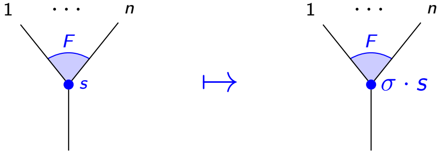
Equivalent notion: symmetric sequences

category $\mathbb{P}$ objects: natural numbers, morphisms: permutations

A **symmetric sequence** is a functor $F : \mathbb{P} \to \mathbf{Set}$ or equivalently

- a set $F(n)$ of F-structures for every $n \in \mathbb{N}$
- a group action $\mathfrak{S}_n \times F(n) \to F(n)$ for every $n \in \mathbb{N}$

$$\begin{tikzcd} F(n) \arrow[r, "F(\sigma)"] & F(n) \end{tikzcd}$$



Kelly, "On the operads of J.P. May", 2005

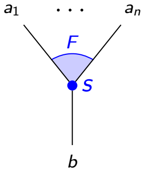
Joyal, "Foncteurs analytiques et espèces de structures", 2006

Colored symmetric sequences

For sets A, B, an **(A, B)-colored symmetric sequence** consists of:

- for every pair $(\langle a_1, \ldots, a_n \rangle, b)$ of a list of input colors $a_1, \ldots, a_n \in A$ and an output color $b \in B$, a set $F(\langle a_1, \ldots, a_n \rangle, b)$

a general element
$s \in F(\langle a_1, \ldots, a_n \rangle, b)$

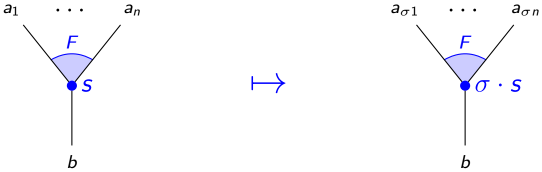


Colored symmetric sequences

For sets A, B, an (A, B)-**colored symmetric sequence** consists of:

- for every pair $(\langle a_1, \dots, a_n \rangle, b)$ of a list of input colors $a_1, \dots, a_n \in A$ and an output color $b \in B$, a set $F(\langle a_1, \dots, a_n \rangle, b)$
- for every permutation $\sigma \in \mathfrak{S}_n$, a bijection satisfying group action axioms

$$F(\langle a_1, \dots, a_n \rangle, b) \xrightarrow{F(\sigma, b)} F(\langle a_{\sigma 1}, \dots, a_{\sigma n} \rangle, b)$$



Méndez, "Colored Species, c-Monoids, and Plethysm, I", 1993

Generalized species of structure

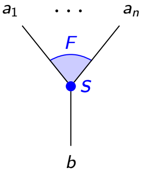
Generalization from a set of colors to a category of colors

Generalized species of structure

For categories $\mathbf{A}, \mathbf{B}$, an **$(\mathbf{A}, \mathbf{B})$-generalized species**, written $F : \mathbf{A} \xrightarrow{E} \mathbf{B}$, consists of:

- for every pair $(\langle a_1, \ldots, a_n \rangle, b)$ of a list of input colors $a_1, \ldots, a_n \in \mathbf{A}$ and an output color $b \in \mathbf{B}$, a set $F(\langle a_1, \ldots, a_n \rangle, b)$

a general element
$s \in F(\langle a_1, \ldots, a_n \rangle, b)$

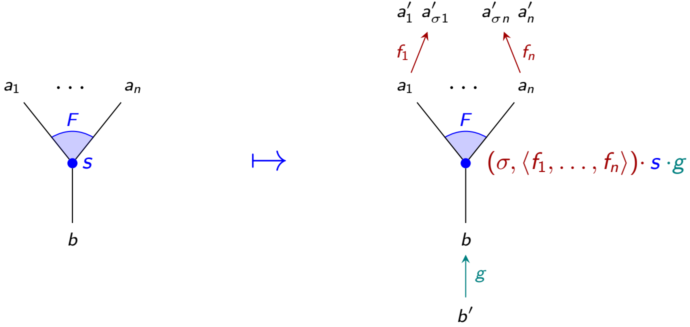


Generalized species of structure

For categories $\mathbf{A}, \mathbf{B}$, an **$(\mathbf{A}, \mathbf{B})$-generalized species**, written $F : \mathbf{A} \xrightarrow{E} \mathbf{B}$, consists of:

- for every permutation $\sigma \in \mathfrak{S}_n$, lists of morphisms $\langle f_1, \dots, f_n \rangle$ with $f_i : a_i \to a'_{\sigma i}$ in $\mathbf{A}$ and morphism $g : b' \to b$ in $\mathbf{B}$ a function satisfying some axioms

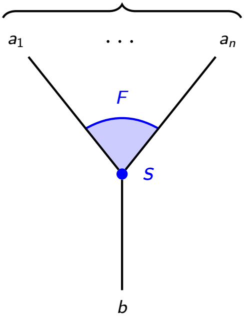
$$F(\langle a_1, \dots, a_n \rangle, b) \xrightarrow{F((\sigma, \langle f_1, \dots, f_n \rangle), g)} F(\langle a'_1, \dots, a'_n \rangle, b')$$



Generalized species of structure

generalized species = many-to-one operations

finite sequence of objects



single object

left action on inputs via permutations and morphisms in the input category

right action on the output via morphisms in the output category

Generalized species of structure

Fiore, “Mathematical Models of Computational and Combinatorial Structures: (Invited Address)”, 2005.

Fiore, Gambino, Hyland, Winskel, “The cartesian closed bicategory of generalised species of structures”, 2008

Earlier influences

- Joyal’s species and their relationship with operads
- Linear logic 80’s: Girard
- Concurrency 90’s: Cattani, Fiore, Moggi, Sangiorgi, Winskel (π-calculus, process algebras)
- Abstract syntax 90’s: Fiore, Plotkin, Turi

Generalized species as functors

- An $(\mathbf{A}, \mathbf{B})$-generalized species $F : \mathbf{A} \xrightarrow{E} \mathbf{B}$ is equivalently a functor

$$\mathcal{S}\mathbf{A} \longrightarrow \widehat{\mathbf{B}}$$

free symmetric strict
monoidal completion

presheaves over $\mathbf{B}$

Generalized species as functors

- An $(\mathbf{A},\mathbf{B})$-generalized species $F : \mathbf{A} \xrightarrow{E} \mathbf{B}$ is equivalently a functor

$$\frac{\mathcal{S}\mathbf{A} \longrightarrow \widehat{\mathbf{B}}}{\mathcal{S}\mathbf{A} \times \mathbf{B}^{\mathrm{op}} \longrightarrow \mathbf{Set}}$$

Generalized species as functors

- An $(\mathbf{A}, \mathbf{B})$-generalized species $F : \mathbf{A} \xrightarrow{E} \mathbf{B}$ is equivalently a functor

$$\frac{\mathcal{S}\mathbf{A} \longrightarrow \widehat{\mathbf{B}}}{\mathcal{S}\mathbf{A} \times \mathbf{B}^{\mathrm{op}} \longrightarrow \mathbf{Set}}$$

The category $\mathcal{S}\mathbf{A}$:

- objects: $\langle a_1, \ldots, a_n \rangle$ finite sequences of objects of $\mathbf{A}$
- morphisms $\langle a_1, \ldots, a_n \rangle \to \langle a'_1, \ldots, a'_n \rangle$ are pairs $(\sigma, \langle f_1, \ldots, f_n \rangle)$ of a permutation $\sigma \in \mathfrak{S}_n$ and a finite sequence of morphisms $f_i : a_i \to a'_{\sigma(i)}$ in $\mathbf{A}$

$$\begin{tikzcd}
\underline{n} \arrow[d, "\sigma"', "\cong"] \\
\underline{n}
\end{tikzcd}
\qquad
\begin{tikzcd}
a_1 \arrow[drr, "f_1"] & \cdots & a_i \arrow[drr, "f_i"] & \cdots & a_n \arrow[dllll, "f_n"'] \\
a'_1 & \cdots & a'_{\sigma(1)} & \cdots & a'_n
\end{tikzcd}$$

Generalized species as functors

- An $(\mathbf{A},\mathbf{B})$-generalized species $F:\mathbf{A}\xrightarrow{E}\mathbf{B}$ is equivalently a functor

$$\frac{\mathcal{S}\mathbf{A}\longrightarrow\widehat{\mathbf{B}}}{\mathcal{S}\mathbf{A}\times\mathbf{B}^{\mathrm{op}}\longrightarrow\mathbf{Set}}$$

Relationship with ordinary species:

- Let $\mathbf{1}$ be the category with a single object and a single morphism
- A $(\mathbf{1},\mathbf{1})$-species of structure corresponds to a symmetric sequence

$$F:\mathcal{S}\,\mathbf{1}\to\widehat{\mathbf{1}}\qquad\Leftrightarrow\qquad F:\mathbb{P}\to\mathbf{Set}$$

Example: species of sets

- Combinatorial species of sets

$$\begin{aligned} E : \mathbb{B} &\to \textbf{Set} \\ A &\mapsto \{\star\} \end{aligned}$$

- Symmetric sequence of sets

$$\begin{aligned} E : \mathbb{P} &\to \textbf{Set} \\ n &\mapsto \{\star\} \end{aligned}$$

- Generalized species of sets

$$\begin{aligned} E_{\mathbf{A},\mathbf{B}} : \mathcal{S}\mathbf{A} &\to \widehat{\mathbf{B}} \\ (\langle a_1, \ldots, a_n \rangle, b) &\mapsto \{\star\} \end{aligned}$$

Example: singleton species

- Combinatorial species of singletons

$$X : \mathbb{B} \to \mathbf{Set}$$

$$A \mapsto \begin{cases} A & \text{if } |A| = 1 \\ \varnothing & \text{otherwise} \end{cases}$$

- Symmetric sequence of singletons

$$X : \mathbb{P} \to \mathbf{Set}$$

$$n \mapsto \begin{cases} \{\star\} & \text{if } n = 1 \\ \varnothing & \text{otherwise} \end{cases}$$

- Generalized species of singletons

$$X_{\mathbf{A}} : \mathcal{S}\mathbf{A} \to \widehat{\mathbf{A}}$$

$$(\langle a_1, \ldots, a_n \rangle, a) \mapsto \begin{cases} \mathbf{A}(a, a_1) & \text{if } n = 1 \\ \varnothing & \text{otherwise} \end{cases}$$

Example: lists (linear orders)

- Combinatorial species of lists

$$\begin{aligned} L : \mathbb{B} &\to \mathbf{Set} \\ A &\mapsto \{f : \underline{n} \xrightarrow{\sim} A\} \quad (n := |A|) \end{aligned}$$

- Symmetric sequence of lists

$$\begin{aligned} L : \mathbb{P} &\to \mathbf{Set} \\ n &\mapsto \mathfrak{S}_n \end{aligned}$$

- Generalized species of lists

$$\begin{aligned} L_{\mathbf{A}} : \mathcal{S}\mathbf{A} &\to \widehat{\mathbf{A}} \\ (\langle a_1, \ldots, a_n \rangle, a) &\mapsto \mathcal{S}\mathbf{A}(\underbrace{\langle a, \ldots, a \rangle}_{n \text{ times}}, \langle a_1, \ldots, a_n \rangle) \end{aligned}$$

Operations on Generalized Species

- Sum of combinatorial species $F, G : \mathbb{B} \to \mathbf{Set}$

$$\begin{aligned} F + G : \mathbb{B} &\to \mathbf{Set} \\ A &\mapsto F(A) \uplus G(A) \end{aligned}$$

- Sum of generalized species $F, G : \mathcal{S}\mathbf{A} \to \widehat{\mathbf{B}}$

$$\begin{aligned} F + G : \mathcal{S}\mathbf{A} &\to \widehat{\mathbf{B}} \\ (\langle a_1, \ldots, a_n \rangle, b) &\mapsto F(\langle a_1, \ldots, a_n \rangle, b) \uplus G(\langle a_1, \ldots, a_n \rangle, b) \end{aligned}$$

Operations on Generalized Species

- Multiplication of combinatorial species $F, G : \mathbb{B} \to \mathbf{Set}$

$$\begin{aligned} F \cdot G : \mathbb{B} &\to \mathbf{Set} \\ A &\mapsto \sum_{C \uplus D = A} F(C) \times G(D) \end{aligned}$$

- Day convolution product of generalized species $F, G : \mathcal{S}\mathbf{A} \to \widehat{\mathbf{B}}$

$$F \cdot G : \mathcal{S}\mathbf{A} \to \widehat{\mathbf{B}}$$

$$(\langle a_1, \ldots, a_n \rangle, b) \mapsto \int^{\vec{c}, \vec{d} \in \mathcal{S}\mathbf{A}} \mathcal{S}\mathbf{A}(\vec{c} \cdot \vec{d}, \langle a_1, \ldots, a_n \rangle) \times F(\vec{c}, b) \times G(\vec{d}, b)$$

We obtain the same fixed point formula for lists (linear orders)

$$L_{\mathbf{A}} \cong 1_{\mathbf{A}} + X_{\mathbf{A}} \cdot L_{\mathbf{A}}$$

Composition of Generalized Species

- Composition (substitution) of combinatorial species $F, G : \mathbb{B} \to \mathbf{Set}$

$$\begin{aligned} G \circ F : \mathbb{B} &\to \mathbf{Set} \\ A &\mapsto \sum_{\Pi \in \mathrm{Part}(A)} G(\Pi) \times \prod_{P \in \Pi} F(P) \end{aligned}$$

- Composition of generalized species $F : \mathcal{S}\mathbf{A} \to \widehat{\mathbf{B}}$ and $G : \mathcal{S}\mathbf{B} \to \widehat{\mathbf{C}}$

$$G \circ F : \mathcal{S}\mathbf{A} \to \widehat{\mathbf{C}}$$

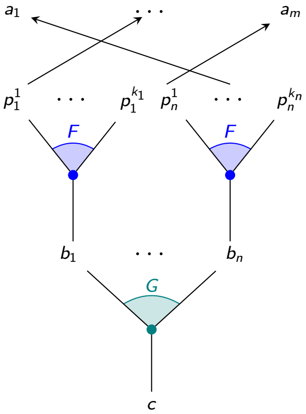
$$(\vec{a}, c) \mapsto \int^{\substack{\vec{b} = \langle b_1, \ldots b_n \rangle \in \mathcal{S}\mathbf{B}, \\ \vec{p}_1, \ldots, \vec{p}_n \in \mathcal{S}\mathbf{A}}} G(\vec{b}, c) \times \prod_{i=1}^{n} F(\vec{p}_i, b_i) \times \mathcal{S}\mathbf{A}(\vec{p}_1 \cdot \cdots \cdot \vec{p}_n, \vec{a})$$

Composition of Generalized Species

$$(G \circ F)(\vec{a}, c) = \int^{\substack{\vec{b}=\langle b_1,\dots b_n\rangle \in \mathcal{S}\mathbf{B}, \\ \vec{p}_1,\dots,\vec{p}_n \in \mathcal{S}\mathbf{A}}} G(\vec{b}, c) \times \prod_{i=1}^{n} F(\vec{p}_i, b_i) \times \mathcal{S}\mathbf{A}(\vec{p}_1 \cdot \dots \cdot \vec{p}_n, \vec{a})$$

A $(G \circ F)(\vec{a}, c)$-structure

is a quotient of elements of the shape:

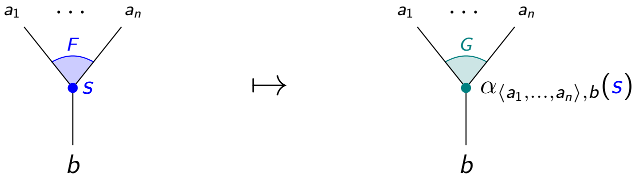


Bicategory of Generalized Species

Generalized species form a **bicategory** **Esp**:

- objects: small categories $\mathbf{A}, \mathbf{B}, \ldots$
- 1-morphisms: generalized species $F : \mathcal{S}\mathbf{A} \to \widehat{\mathbf{B}}$
- 2-morphisms: natural transformations $\alpha : F \Rightarrow G : \mathcal{S}\mathbf{A} \to \widehat{\mathbf{B}}$

$$F(\langle a_1, \ldots, a_n \rangle, b) \xrightarrow{\alpha_{\langle a_1, \ldots, a_n \rangle, b}} G(\langle a_1, \ldots, a_n \rangle, b)$$



family of functions compatible with the left and right actions

Generating series and analytic functors

- Recall, a combinatorial species induces a **generating series**

$$F : \mathbb{B} \to \mathbf{Set} \qquad \rightsquigarrow \qquad \sum_{n \geq 0} |F(\underline{n})| \frac{x^n}{n!}$$

- These series do not capture the action of F on bijections (e.g. the species of lists and permutations have the same generating series)
- **Analytic functors:** notion of functorial series fully capturing the behavior of species

  Joyal, "Foncteurs analytiques et espèces de structures", 2006

- **Generalized analytic functors:** analogue for generalized species

  Fiore, "Analytic functors between presheaf categories over groupoids", 2014

Implicit species theorem

Joyal, “Une théorie combinatoire des séries formelles”, 1981

Implicit species theorem: For a 2-sorted species H verifying

$$H(0,0) = 0 \qquad \text{and} \qquad \frac{\partial H(X,Y)}{\partial Y}(0,0) = 0$$

then there exists a species (unique up to isomorphism) such that

$$H(F, \mathrm{id}) \cong F \qquad \text{and} \qquad F(0) = 0$$

Initial conditions exclude simples cases such as

- lists or linear orderings: $L \cong 1 + X \times L$
- binary rooted trees: $B \cong 1 + X \times B^2$

We can weaken the initial conditions if we consider species $F : \mathbb{B} \to \mathbf{Set}$ and not $F : \mathbb{B} \to \mathbf{FinSet}$.

Implicit species theorem and fixpoint operators

In generalized species, every implicit equation has a solution

- Given a 2-sorted species $H : \mathbf{X} \times \mathbf{Y} \xrightarrow{E} \mathbf{Y}$, a solution is a species $F : \mathbf{X} \xrightarrow{E} \mathbf{Y}$ such that

$$H \circ \langle F, \mathrm{id}_{\mathbf{Y}} \rangle \cong F \qquad \text{“}H(F(Y), Y) = F(Y)\text{''}$$

- We have a **least fixpoint operator**

$$\mathbf{fix} : \mathbf{Esp}(\mathbf{X} \times \mathbf{Y}, \mathbf{Y}) \quad \longrightarrow \quad \mathbf{Esp}(\mathbf{X}, \mathbf{Y})$$

  and we can obtain uniqueness (up to unique isomorphism) via uniformity.

- If we restrict to combinatorial species, **fix** computes the same solution as the one in Joyal's theorem.

G., “Fixpoint operators for 2-categorical structures”, 2023.

There is more

We have seen so far that many constructions on Joyal's species can be generalized to category-colored species.

We will see now that the bicategory of generalized species has also a deep connection to a long line of research on quantitative denotational semantics and linear logic.

Curry-Howard-Lambek correspondence

programming languages ⟺ logic ⟺ category theory

Curry-Howard-Lambek correspondence

simply typed λ-calculus	$\Longleftrightarrow$	intuitionistic logic	$\Longleftrightarrow$	Cartesian closed category
type A		formula A		object $[\![A]\!]$
program $\Gamma \vdash t : B$		proof $\dfrac{\pi}{\Gamma \vdash B}$		morphism $[\![\Gamma]\!] \to [\![B]\!]$
function type		implication		cartesian closed structure
$\lambda x.(x + x) : \mathbb{N} \Rightarrow \mathbb{N}$		proof of $\vdash \mathbb{N} \Rightarrow \mathbb{N}$		morphism $1 \to (\mathbb{N} \Rightarrow \mathbb{N})$

Curry-Howard-Lambek correspondence

simply typed λ-calculus	$\Longleftrightarrow$	intuitionistic logic	$\Longleftrightarrow$	Cartesian closed category
type A		formula A		object $\llbracket A \rrbracket$
program $\Gamma \vdash t : B$		proof $\dfrac{\pi}{\Gamma \vdash B}$		morphism $\llbracket \Gamma \rrbracket \to \llbracket B \rrbracket$

Soundness: invariance under program evaluation/proof rewritings

for any $t \overset{\text{evaluation}}{\rightsquigarrow} t'$ we have $\llbracket t \rrbracket = \llbracket t' \rrbracket$

for any $\dfrac{\pi}{\Gamma \vdash \Delta} \overset{\text{rewriting}}{\rightsquigarrow} \dfrac{\pi'}{\Gamma \vdash \Delta}$ we have $\llbracket \pi \rrbracket = \llbracket \pi' \rrbracket$

Quantitative semantics of linear logic

Girard (1980's): from quantitative semantics ...

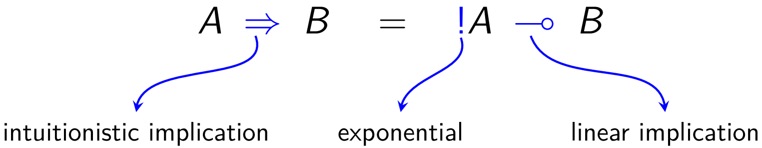
- Type A: vector space
- Program $P : A \Rightarrow B$: analytic map given by a power series

$$[\![P]\!](x) = \sum_{n=0}^{+\infty} P_n \cdot x^n$$

- n: number of times P uses the argument x
- P_n: weight

- Particular case: a program that uses its argument once corresponds to a linear map

... to linear logic:



$$A \Rightarrow B \quad = \quad !A \multimap B$$

Quantitative Semantics and Linear Logic

proof nets $\Longleftrightarrow$ linear logic $\Longleftrightarrow$ $*$-autonomous category + extra structure

Linear logic and quantitative models to account for:

- resource usage (time and space)
- number of ways to obtain a result in a non-deterministic setting
- probability to obtain a result in a probabilistic setting

differential proof nets $\Longleftrightarrow$ differential linear logic $\Longleftrightarrow$ differential category + extra structure

Ehrhard, Regnier (2003):

- differential λ-calculus: notion of Taylor expansion for a program as the sums of its nth linear approximants
- differential linear logic: logical account of differentiation

Derivative of generalized species

Recall, for a species $F : \mathbb{B} \to \mathbf{Set}$,

$$F'(A) = F(A \uplus \{\star\}) \qquad \left(\sum_{n \geq 0} f_n \frac{x^n}{n!} \right)' = \sum_{m \geq 0} f_{m+1} \frac{x^m}{m!}$$

In models of differential logic, the idea is

smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$ $\quad \rightsquigarrow \quad$ Jacobian $\mathbf{J}(f) : \mathbb{R}^n \to \mathbf{Lin}(\mathbb{R}^n, \mathbb{R}^m)$

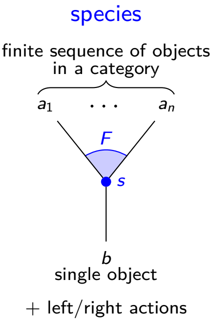
Definition

For a generalized species $F : \mathbf{A} \xrightarrow{E} \mathbf{B}$, its *differential* is the species:

$$\mathbf{D}(F) : \mathbf{A} \xrightarrow{E} (\mathbf{A} \multimap \mathbf{B})$$
$$(\langle a_1, \ldots, a_n \rangle, a, b) \mapsto F(\langle a_1, \ldots, a_n \rangle \cdot \langle a \rangle, b)$$

Species as a bicategorical model of differential linear logic

The bicategory of generalized species is cartesian closed and is closely related to the (weighted) relational model of differential linear logic

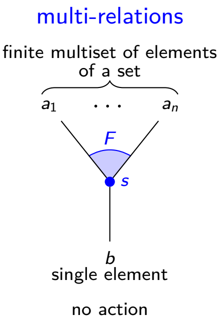


species

finite sequence of objects in a category

single object

+ left/right actions



multi-relations

finite multiset of elements of a set

single element

no action

- Fiore, Gambino, Hyland, Winskel, "The cartesian closed bicategory of generalised species of structures", 2008
- Fiore, Gambino, Hyland, "Monoidal bicategories, differential linear logic, and analytic functors", 2025

Bicategorical semantics

Use 2-morphisms in a bicategory to model computation steps

calculus	bicategory
type/formula A	object $\llbracket A \rrbracket$
term/proof $\dfrac{\pi}{\Gamma \vdash \Delta}$	1-morphism $\llbracket \pi \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow \llbracket \Delta \rrbracket$
$\dfrac{\pi}{\Gamma \vdash \Delta} \underset{\text{computation}}{\rightsquigarrow} \dfrac{\pi'}{\Gamma \vdash \Delta}$	2-morphism $\begin{tikzcd} \llbracket \Gamma \rrbracket \arrow[r, bend left, "\llbracket \pi \rrbracket"] \arrow[r, bend right, "\llbracket \pi' \rrbracket"'] \arrow[r, phantom, "\Downarrow"] & \llbracket \Delta \rrbracket \end{tikzcd}$

What are the research directions today?

type theory

intersection types

abstract syntax

denotational
semantics

differential
linear logic

λ-calculus

game semantics

dualities

combinatorics

Joyal's species

generalized species

2-dimensional
category theory

distributive laws

relative
pseudo-monads

∞-categories

opetopes

polynomial functors

formal languages

automata

operads

Bicategorical denotational semantics

- Tsukada, Asada, Ong, "Generalised species of rigid resource terms", 2017
- Tsukada, Asada, Ong, "Species, profunctors and Taylor expansion weighted by SMCC", 2018

- Taylor expansion of a program is a species such that

$$M \xrightarrow{\text{evaluation}} N \qquad \Rightarrow \qquad \mathrm{Taylor}(M) \xRightarrow{\text{isomorphism}} \mathrm{Taylor}(N)$$

- Weighted generalized species: the weights form a symmetric monoidal closed category and different weight categories induce models for nondeterministic, probabilistic and quantum programs.

Proof-relevant intersection types

- In intersection typing systems, typability is equivalent to termination
- There are many quantitative variants to characterize more refined properties

relations

$$[\![M]\!] = \{(\Gamma, A) \mid \Gamma \vdash M : A\}$$

$$M \xrightarrow{\text{evaluation}} N \quad \Rightarrow \quad [\![M]\!] = [\![N]\!]$$

species

$$[\![M]\!](\Gamma, A) \; = \; \left\{ \begin{array}{c} \pi \\ \vdots \\ \hline \Gamma \vdash M : A \end{array} \right\}$$

$$M \xrightarrow{\text{evaluation}} N \quad \Rightarrow \quad [\![M]\!] \overset{\text{iso}}{\Longrightarrow} [\![N]\!]$$

Olimpieri, “Intersection type distributors.”, 2021

Kerinec, Manzonetto, Olimpieri. “Why are proofs relevant in proof-relevant models?”, 2023

Generalized species and dualities

- Orthogonality categories: tool to refine a model based on a duality between points an co-points while preserving the structure

G. "A bicategorical model for finite nondeterminism", 2021

- Duality on the action of permutation to connect species and polynomial functors (another categorical notion of series)

Fiore, G., Paquet, "A combinatorial approach to higher-order structure for polynomial functors", 2022

Also related to dualities in game semantics...

Bicategorical game semantics

- Bicategorical game models are finer than species: they are quantitative and also take into account the interactive behavior of programs
- Study the relationship between the two models (in a compositional way)

Clairambault, Olimpieri, Paquet, “From thin concurrent games to generalized species of structures”, 2023

Clairambault, Forest, “An analysis of symmetry in quantitative semantics”, 2024

Species and operads

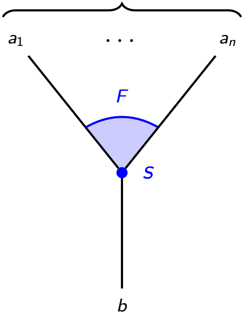
symmetric operads May, 1972	monoids in symmetric sequences with substitution Kelly, 2005
set-colored symmetric operads Boardman, Vogt 1973	monoids in colored symmetric sequences with substitution ?
category-colored symmetric operads Baez, Dolan 1997	internal monads in the bicategory of generalized species Fiore, Gambino, Hyland, Winskel 2008

+ generalizations to the enriched setting (cf. substitudes)

Species, operads, monads and generalized multicategories

generalized species $F : \mathbf{A} \xrightarrow{E} \mathbf{B}$

object of the free symmetric strict monoidal completion 2-monad on **A**



single object of **B**

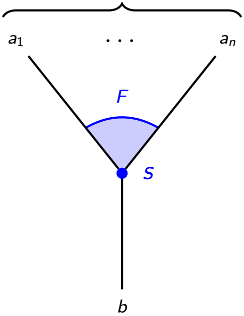
left action on inputs via **permutations** and morphisms in the input category **A**

right action on the output via morphisms in the output category **B**

monoids $=$ **symmetric** operads

Species, operads, monads and generalized multicategories

cartesian species $F : \mathbf{A} \xrightarrow{E} \mathbf{B}$



object of the free cocartesian completion 2-monad on $\mathbf{A}$

single object of $\mathbf{B}$

left action on inputs via **functions** and morphisms in the input category $\mathbf{A}$

right action on the output via morphisms in the output category $\mathbf{B}$

monoids $=$ **cartesian** operads (clones)

Species, operads, monads and generalized multicategories

Two related approaches:

- Generalized multicategories: which 2-monads induce a nice notion of multi-category (=colored operad)?

Cruttwell, Shulman, “A unified framework for generalized multicategories”, 2009

- Monad distributive laws: which 2-monads interact nicely with presheaves?

Fiore, Gambino, Hyland, Winskel, “Relative pseudomonads, Kleisli bicategories, and substitution monoidal structures”, 2018

Hyland, Tasson, “The linear-non-linear substitution 2-monad”, 2020

2-dimensional category theory

Generalized species were an important example for the development of the theory of relative pseudomonads, strong and monoidal pseudomonads

- Fiore, Gambino, Hyland, Winskel, “Relative pseudomonads, Kleisli bicategories, and substitution monoidal structures”, 2018
- Slattery, “Pseudocommutativity and lax idempotency for relative pseudomonads”, 2023
- Miranda, “Eilenberg-Moore bicategories for opmonoidal pseudomonads”, 2024
- Paquet, Saville, “Effectful semantics in bicategories: strong, commutative, and concurrent pseudomonads”, 2024
- Arkor, Saville, Slattery, “Bicategories of algebras for relative pseudomonads”, 2025

Homotopy theory, ∞-categories and opetopes

- Kock, “Data types with symmetries and polynomial functors over groupoids”, 2012
- Fiore, “An Algebraic Combinatorial Approach to the Abstract Syntax of Opetopic Structures”, 2016
- Finster, Mimram, Lucas, Seiller, “A cartesian bicategory of polynomial functors in homotopy type theory”, 2021
- Gepner, Haugseng, Kock “∞-operads as analytic monads”, 2022
- Harington, Mimram, “Polynomials in homotopy type theory as a Kleisli category”, 2024

Formal languages

- Melliès, Zeilberger, “Parsing as a lifting problem and the Chomsky-Schützenberger representation theorem”, 2022
- Melliès, Zeilberger,“The categorical contours of the Chomsky-Schützenberger representation theorem”, 2023
- Loregian,“Automata and coalgebras in categories of species”, 2024

Thank you