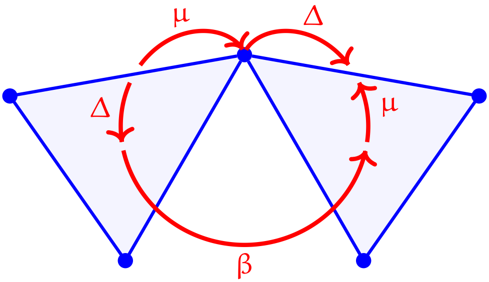
Hopf monoids relative to a hyperplane arrangement



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March 16, 2026

Part I

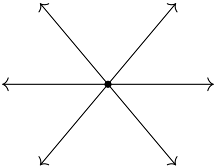
HYPERPLANE ARRANGEMENTS

Real hyperplane arrangements

Let V be a finite-dimensional vector space over $\mathbb{R}$.
Let $\mathscr{A}$ be a finite set of hyperplanes in V.

- The hyperplanes in $\mathscr{A}$ split V into a collection $\Sigma(\mathscr{A})$ of convex sets called faces.
- The subspaces obtained as intersections of hyperplanes in $\mathscr{A}$ are called flats. Let $\Pi(\mathscr{A})$ be the set of flats.

3 lines, 13 faces, 5 flats.

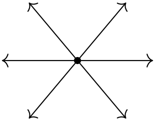


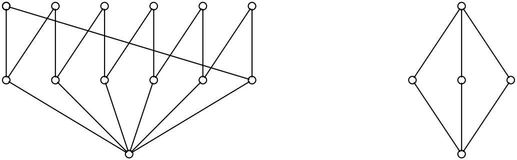
Faces and flats

- The sets $\Sigma(\mathscr{A})$ and $\Pi(\mathscr{A})$ are partially ordered by inclusion.
- Each face is supported on a flat. The support map

$$\mathrm{s} : \Sigma(\mathscr{A}) \to \Pi(\mathscr{A})$$

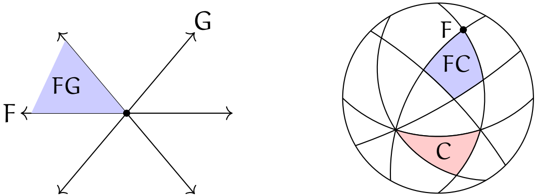
is order-preserving.





Faces and flats

- The set $\Sigma(\mathscr{A})$ is a monoid under the Tits product.



- The set $\Pi(\mathscr{A})$ is a commutative monoid.
- The support map is abelianization:

$$\mathrm{s} : \Sigma(\mathscr{A}) \to \Pi(\mathscr{A}).$$

The braid arrangement

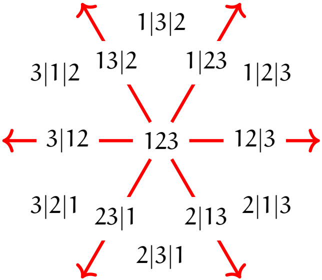
The braid arrangement is the collection of hyperplanes

$$x_i = x_j \quad \text{in } \mathbb{R}^n.$$

- Faces are in bijection with ordered partitions of $[n]$,

$$\text{e.g. } 1|23 = \{(x_1, x_2, x_3) \mid x_1 > x_2 = x_3\}.$$

- Flats are in bijection with partitions of $[n]$.



1|3|2

3|1|2 13|2 1|23 1|2|3

3|12 123 12|3

3|2|1 23|1 2|13 2|1|3

2|3|1

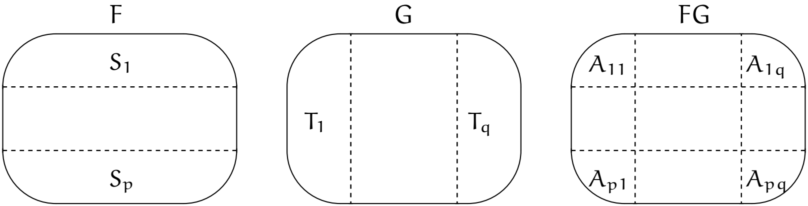
The Tits product for the braid arrangement

Let I be a finite set. Let $\mathscr{B}^{\mathrm{I}}$ be the braid arrangement in $\mathbb{R}^{\mathrm{I}}$.

- $\Sigma(\mathscr{B}^{\mathrm{I}})$ is the set of ordered partitions F of I:

  $$\mathrm{F} = (S_1, \ldots, S_p) \quad \mathrm{I} = S_1 \cup \cdots \cup S_p, \;\; S_i \neq \emptyset, \;\; S_i \cap S_j = \emptyset \text{ if } i \neq j.$$

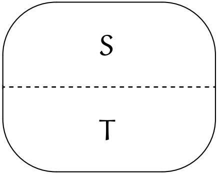
- $\Pi(\mathscr{B}^{\mathrm{I}})$ is the set of partitions of I.
- $\mathrm{s} : \Sigma(\mathscr{B}^{\mathrm{I}}) \to \Pi(\mathscr{B}^{\mathrm{I}})$ forgets the order among the blocks.
- The Tits product is lexicographic refinement.

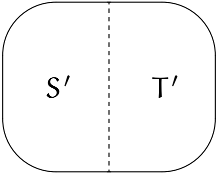


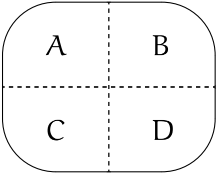
(remove empty A_{ij}'s)

Noncommutativity and middle interchange

Let $F = (S, T)$ and $G = (S', T')$ be two ordered partitions of I.







Then

$$FG = (A, B, C, D)$$

$$GF = (A, C, B, D)$$

Part II

HOPF MONOIDS

Species relative to a hyperplane arrangement

Fix a hyperplane arrangement $\mathscr{A}$.
An $\mathscr{A}$-species p is a family of vector spaces $\mathsf{p}[F]$, one for each face F, with linear maps

$$\beta_{G,F} : \mathsf{p}[F] \to \mathsf{p}[G],$$

whenever F and G have the same support, such that

$$(\mathsf{p}[F] \xrightarrow{\beta_{F,F}} \mathsf{p}[F]) = \mathrm{id}$$

and

$$\begin{tikzcd} & \mathsf{p}[G] \arrow[dr, "\beta_{H,G}"] & \\ \mathsf{p}[F] \arrow[ur, "\beta_{G,F}"] \arrow[rr, "\beta_{H,F}"'] & & \mathsf{p}[H] \end{tikzcd}$$

whenever F, G and H have the same support.

Monoids relative to a hyperplane arrangement

An $\mathscr{A}$-monoid is an $\mathscr{A}$-species m with linear maps

$$\mu_A^F : \mathrm{m}[F] \to \mathrm{m}[A],$$

one for each $A \leq F$, such that

$$\begin{tikzcd}
\mathrm{m}[F] \arrow[r, "\beta_{BF,F}"] \arrow[d, "\mu_A^F"'] & \mathrm{m}[BF] \arrow[d, "\mu_B^{BF}"] \\
\mathrm{m}[A] \arrow[r, "\beta_{B,A}"'] & \mathrm{m}[B]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{m}[F] \arrow[dr, "\mu_A^F"] & \\
\mathrm{m}[G] \arrow[ur, "\mu_F^G"] \arrow[rr, "\mu_A^G"'] & & \mathrm{m}[A]
\end{tikzcd}
\qquad
\left(\mathrm{m}[A] \xrightarrow{\mu_A^A} \mathrm{m}[A]\right) = \mathrm{id}\,.$$

$(A \sim B$ and $A \leq F)$

$(A \leq F \leq G)$

The $\mathscr{A}$-monoid (m, μ) is commutative if

$$\begin{tikzcd}
\mathrm{a}[F] \arrow[rr, "\beta_{G,F}"] \arrow[dr, "\mu_A^F"'] & & \mathrm{a}[G] \arrow[dl, "\mu_A^G"] \\
 & \mathrm{a}[A] &
\end{tikzcd}$$

$(A \leq F$, $A \leq G$, and $F \sim G)$.

Comonoids relative to a hyperplane arrangement

An $\mathscr{A}$-comonoid is an $\mathscr{A}$-species c with linear maps

$$\Delta_A^F : \mathsf{c}[A] \to \mathsf{c}[F]$$

one for each $A \leq F$, satisfying dual axioms.

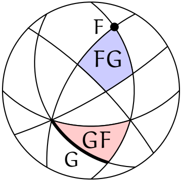
Bimonoids relative to a hyperplane arrangement

An $\mathscr{A}$-bimonoid is an $\mathscr{A}$-species h with linear maps

$$\begin{tikzcd} \mathrm{h}[F] \arrow[r, shift left, "\mu_A^F"] & \mathrm{h}[A] \arrow[l, shift left, "\Delta_A^F"] \end{tikzcd}$$

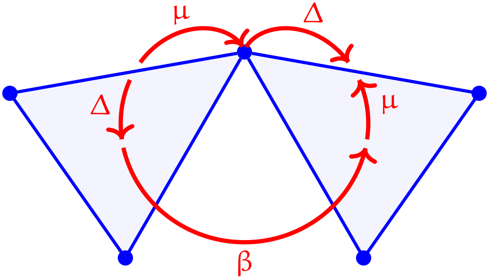
such that (h, μ) is an $\mathscr{A}$-monoid, (h, Δ) is an $\mathscr{A}$-comonoid, and for any faces $A \leq F$ and $A \leq G$,

$$\begin{tikzcd} \mathrm{h}[F] \arrow[r, "\mu_A^F"] \arrow[d, "\Delta_F^{FG}"'] & \mathrm{h}[A] \arrow[r, "\Delta_A^G"] & \mathrm{h}[G] \\ \mathrm{h}[FG] \arrow[rr, "\beta_{GF,FG}"'] & & \mathrm{h}[GF]. \arrow[u, "\mu_G^{GF}"'] \end{tikzcd}$$



$A \leq F$ and $A \leq G$,

$$\begin{tikzcd}
h[F] \arrow[r, "\mu_A^F"] \arrow[d, "\Delta_F^{FG}"'] & h[A] \arrow[r, "\Delta_A^G"] & h[G] \\
h[FG] \arrow[rr, "\beta_{GF,FG}"'] & & h[GF]. \arrow[u, "\mu_G^{GF}"']
\end{tikzcd}$$



Hopf monoids relative to a hyperplane arrangement

There is no distinction between $\mathscr{A}$-bimonoids and $\mathscr{A}$-Hopf monoids.

The antipode of h is the family of linear maps

$$S_A : \mathsf{h}[A] \to \mathsf{h}[A]$$

given by

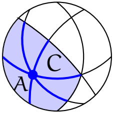
$$S_A = \sum_{F:\, A \leq F} (-1)^{\mathrm{rk}(F)} \mu_A^F \Delta_A^F.$$

Example: the Hopf monoid of chambers

The chambers of $\mathscr{A}$ are its top-dimensional faces.

Define an $\mathscr{A}$-species h by

$$\mathsf{h}[A] = \Bbbk\{C \mid C \text{ is a chamber and } A \le C\}.$$



Given $A \le F$, define linear maps

$$\mathsf{h}[F] \underset{\Delta_A^F}{\overset{\mu_A^F}{\rightleftarrows}} \mathsf{h}[A] \quad \text{by} \quad \mu_A^F(C) = C \quad \text{and} \quad \Delta_A^F(C) = FC.$$

Then $(\mathsf{h}, \mu, \Delta)$ is an $\mathscr{A}$-Hopf monoid. Moreover,

$$S_A(C) = (-1)^{\mathrm{rk}(\mathscr{A})}(\text{the opposite to } C \text{ across } A).$$

In particular,

$$S_O(C) = (-1)^{\mathrm{rk}(\mathscr{A})}(\text{the antipodal chamber of } C).$$

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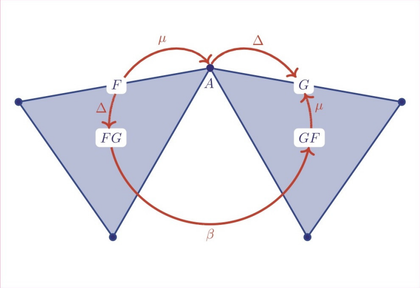
BIMONOIDS FOR HYPERPLANE ARRANGEMENTS

Marcelo Aguiar and Swapneel Mahajan



Part III

HOPF MONOIDS and HOPF ALGEBRAS

Joyal's species

A Joyal species is a functor

$$\mathsf{p} : \mathsf{set}^{\times} \to \mathsf{Vec}.$$

For each finite set I, $\mathsf{p}[I]$ is a vector space.

Fix I. Given an ordered partition $F = (S_1, \ldots, S_p)$ of I, define

$$\mathsf{p}^I[F] = \mathsf{p}[S_1] \otimes \cdots \otimes \mathsf{p}[S_p].$$

If $\mathrm{s}(F) = \mathrm{s}(G)$, there is an isomorphism

$$\beta_{G,F} : \mathsf{p}^I[F] \to \mathsf{p}^I[G],$$

defined by rearranging the tensor factors.

Hence, p^I is a $\mathscr{B}^I$-species.

Hopf monoids in Joyal's species

The Cauchy product of two Joyal species p and q is

$$(\mathsf{p} \cdot \mathsf{q})[I] = \bigoplus_{\substack{I=S\cup T \\ \emptyset=S\cap T}} \mathsf{p}[S] \otimes \mathsf{q}[T].$$

The category of Joyal species is symmetric monoidal.

We can speak of monoids, comonoids,. . . , in Joyal's species.

$$\begin{tikzcd} \mathsf{h}[S] \otimes \mathsf{h}[T] \arrow[r, shift left, "\mu_{S,T}"] & \mathsf{h}[I] \arrow[l, shift left, "\Delta_{S,T}"] \end{tikzcd}$$

We assume $\mathsf{h}[\emptyset] = \mathbb{k}$.
Then, there is no difference between bimonoids and Hopf monoids.

Example: the Hopf monoid of linear orders

Let $\mathsf{L}[I]$ be the set of all linear orders on I.

$$\mathsf{L}[a,b,c] = \{abc,\ bac,\ acb,\ bca,\ cab,\ cba\}.$$

The product $\Bbbk\,\mathsf{L}[S] \otimes \Bbbk\,\mathsf{L}[T] \to \Bbbk\,\mathsf{L}[I]$ is concatenation:
if $\ell_1 = s_1 \dots s_i \in \mathsf{L}[S]$ and $\ell_2 = t_1 \dots t_j \in \mathsf{L}[T]$, then
$\mu_{S,T}(\ell_1 \otimes \ell_2) = s_1 \dots s_i\, t_1 \dots t_j$.

The coproduct $\Bbbk\,\mathsf{L}[I] \to \Bbbk\,\mathsf{L}[S] \otimes \Bbbk\,\mathsf{L}[T]$ is restriction:
if $\ell \in \mathsf{L}[I]$, then $\Delta_{S,T}(\ell) = \ell|_S \otimes \ell|_T$.

Then $\Bbbk\,\mathsf{L}$ is a Hopf monoid. Moreover, the antipode is reversal:
if $\ell = a_1 \dots a_n \in \mathsf{L}[I]$, then $S_I(\ell) = (-1)^n a_n \dots a_1$,

Connection to $\mathscr{B}^{\mathrm{I}}$-Hopf monoids

Let h be a Hopf monoid in Joyal's category of species.

Recall that for an ordered partition $F = (S_1, \ldots, S_p)$ of I,

$$\mathsf{h}^{I}[F] = \mathsf{h}[S_1] \otimes \cdots \otimes \mathsf{h}[S_p].$$

There are maps

$$\begin{tikzcd} \mathsf{h}^{I}[F] \arrow[r, shift left, "\mu_{S_1,\ldots,S_p}"] & \mathsf{h}[I] \arrow[l, shift left, "\Delta_{S_1,\ldots,S_p}"] \end{tikzcd}$$

well-defined by (co)associativity.

More generally, given ordered partitions $A \leq F$, there are maps

$$\begin{tikzcd} \mathsf{h}^{I}[F] \arrow[r, shift left, "\mu_A^F"] & \mathsf{h}^{I}[A] \arrow[l, shift left, "\Delta_A^F"] \end{tikzcd}$$

defined by taking tensor products of the maps above.

In this manner, h^{I} is a $\mathscr{B}^{I}$-Hopf monoid.

Connection to Hopf algebras

Theorem. There is a functor

$$\{\text{Hopf monoids in Joyal's species}\} \xrightarrow{\mathscr{K}} \{\mathbb{N}\text{-graded Hopf algebras}\}$$

given on a Hopf monoid h by

$$\mathscr{K}(\mathrm{h})_n = \mathrm{h}[n]^{S_n}.$$

This functor has a left adjoint $\mathscr{L}$ and a right adjoint $\mathscr{R}$ such that

$$\mathscr{K}\mathscr{L}(\mathrm{H}) \cong \mathrm{H} \cong \mathscr{K}\mathscr{R}(\mathrm{H})$$

for any connected graded Hopf algebra H.

The bimonoid axiom

$$\begin{tikzcd}
h \cdot h \arrow[r, "\mu"] \arrow[d, "\Delta \cdot \Delta"'] & h \arrow[r, "\Delta"] & h \cdot h \\
h \cdot h \cdot h \cdot h \arrow[rr, "\mathrm{id} \cdot \beta \cdot \mathrm{id}"'] & & h \cdot h \cdot h \cdot h \arrow[u, "\mu \cdot \mu"']
\end{tikzcd}$$

Symmetric monoidal category

$$\begin{tikzcd}
h[S] \otimes h[T] \arrow[r, "\mu_{S,T}"] \arrow[d, "\Delta_{A,B} \otimes \Delta_{C,D}"'] & h[I] \arrow[r, "\Delta_{S',T'}"] & h[S'] \otimes h[T'] \\
h[A] \otimes h[B] \otimes h[C] \otimes h[D] \arrow[rr, "\mathrm{id}_A \otimes \beta_{B,C} \otimes \mathrm{id}_D"'] & & h[A] \otimes h[C] \otimes h[B] \otimes h[D] \arrow[u, "\mu_{A,C} \otimes \mu_{B,D}"']
\end{tikzcd}$$

Joyal's species

$$\begin{tikzcd}
h[F] \arrow[r, "\mu_O^F"] \arrow[d, "\Delta_F^{FG}"'] & h[O] \arrow[r, "\Delta_O^G"] & h[G] \\
h[FG] \arrow[rr, "\beta_{GF,FG}"'] & & h[GF]. \arrow[u, "\mu_G^{GF}"']
\end{tikzcd}$$

Relative to an arrangement

$$F = (S, T),\ G = (S', T') \Rightarrow FG = (A, B, C, D),\ GF = (A, C, B, D)$$

Example

Let $\mathsf{h} = \Bbbk\,\mathsf{L}$ be the Hopf monoid of linear orders in Joyal's species.

$\mathscr{K}(\mathsf{h})_n = \mathsf{h}[n]^{S_n}$ is one-dimensional, and

$$\mathscr{K}(\mathsf{h}) = \Bbbk[x],$$

the Hopf algebra of polynomials.

Recall: the chambers of $\mathscr{B}^I$ correspond to linear orders on I.
h^I is the $\mathscr{B}^I$-Hopf monoid of chambers.

Hopf monoid of chambers	Relative to an arrangement
Hopf monoid of linear orders	Joyal's species
Hopf algebra $\Bbbk[x]$	Vector spaces

Part IV

HOPF THEORY

Hopf powers

Let h be an $\mathscr{A}$-Hopf monoid.
Given a face F of $\mathscr{A}$, consider the linear map

$$\mu_O^F \Delta_O^F : \mathrm{h}[O] \to \mathrm{h}[O].$$

Proposition.
If h is cocommutative, this is a left action of $\Sigma(\mathscr{A})$ on $\mathrm{h}[O]$.
If h is commutative, this is a right action.

When $\mathscr{A} = \mathscr{B}^I$, $F = (S_1, \ldots, S_p)$, these maps are

$$\mu_{S_1,\ldots,S_p} \Delta_{S_1,\ldots,S_p} : \mathrm{h}[I] \to \mathrm{h}[I].$$

Analogous to Hopf powers of a Hopf algebra H

$$\mu^{(p-1)} \Delta^{(p-1)} : H \to H.$$

The Tits algebra

Let $\Bbbk\Sigma(\mathscr{A})$ be the algebra of the monoid $\Sigma(\mathscr{A})$.

Theorem. There is a category equivalence

$$\begin{aligned}\{\text{cocommutative } \mathscr{A}\text{-Hopf monoids}\} &\simeq \{\text{left } \Bbbk\Sigma(\mathscr{A})\text{-modules}\}.\\ \mathrm{h} &\mapsto \mathrm{h}[O]\end{aligned}$$

Proof. Let $\mathrm{K}(\mathscr{A})$ be the Karoubi envelope of the monoid $\Sigma(\mathscr{A})$. It is a category. We have equivalences:

- $\Bbbk\Sigma(\mathscr{A})$-modules $\simeq [\Sigma(\mathscr{A}), \mathsf{Vec}_{\Bbbk}]$,
- $[\Sigma(\mathscr{A}), \mathsf{Vec}_{\Bbbk}] \simeq [\mathrm{K}(\mathscr{A}), \mathsf{Vec}_{\Bbbk}]$,
- $[\mathrm{K}(\mathscr{A}), \mathsf{Vec}_{\Bbbk}] \simeq$ cocommutative $\mathscr{A}$-Hopf monoids.

Similarly, there are equivalences

$$\{\text{commutative } \mathscr{A}\text{-Hopf monoids}\} \simeq \{\text{right } \Bbbk\Sigma(\mathscr{A})\text{-modules}\},$$

$$\{\text{bicommutative } \mathscr{A}\text{-Hopf monoids}\} \simeq \{\Bbbk\Pi(\mathscr{A})\text{-modules}\}.$$

The primitive filtration

Let c be an $\mathscr{A}$-comonoid. Define its primitive part $\mathscr{P}(\mathsf{c})$ by

$$\mathscr{P}(\mathsf{c})[A] = \bigcap_{F:\, F>A} \ker(\Delta_A^F : \mathsf{c}[A] \to \mathsf{c}[F]).$$

More generally, for $k \geq 1$, define $\mathscr{P}_k(\mathsf{c})$ by

$$\mathscr{P}_k(\mathsf{c})[A] = \bigcap_{\substack{F:\, F\geq A,\\ \mathrm{rk}(F/A)\geq k}} \ker(\Delta_A^F : \mathsf{c}[A] \to \mathsf{c}[F]).$$

We have

$$\mathscr{P}_1(\mathsf{c}) \subseteq \mathscr{P}_2(\mathsf{c}) \subseteq \cdots \subseteq \mathsf{c} \quad \text{and} \quad \bigcup_{k\geq 1} \mathscr{P}_k(\mathsf{c}) = \mathsf{c}.$$

Theorem. Let h be a cocommutative $\mathscr{A}$-Hopf monoid.
Then $\mathrm{gr}_{\mathscr{P}}(\mathsf{h})$ is bicommutative.
Proof.

- The primitive filtration is a Loewy series for the Tits algebra.
- $\Bbbk\Pi(\mathscr{A})$ is the largest semisimple quotient of $\Bbbk\Sigma(\mathscr{A})$.

Structure theorems

Let h be an $\mathscr{A}$-Hopf monoid.

- Leray-Samelson: If h is bicommutative, then

  $$\mathsf{h} \cong \mathscr{S}\mathscr{P}(\mathsf{h})$$

  as Hopf monoids.
- Borel-Hopf: If h is cocommutative, then

  $$\mathsf{h} \cong \mathscr{S}\mathscr{P}(\mathsf{h})$$

  as comonoids.
- Cartier-Milnor-Moore: If h is cocommutative, then

  $$\mathsf{h} \cong \mathscr{U}\mathscr{P}(\mathsf{h})$$

  as Hopf monoids.

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Lie monoids relative to a hyperplane arrangement

An $\mathscr{A}$-Lie monoid is a species g with linear maps

$$\nu^{F}_{A} : g[F] \longrightarrow g[A] \qquad (A \lessdot F),$$

such that:

$$\begin{tikzcd} g[F] \arrow[r, "\beta_{BF,F}"] \arrow[d, "\nu^{F}_{A}"'] & g[BF] \arrow[d, "\nu^{BF}_{B}"] \\ g[A] \arrow[r, "\beta_{B,A}"'] & g[B] \end{tikzcd} \qquad (B \sim A \lessdot F),$$

$$\left(g[F] \xrightarrow{\nu^{F}_{A}} g[A]\right) \; + \; \left(g[F] \xrightarrow{\beta_{A\overline{F},F}} g[A\overline{F}] \xrightarrow{\nu^{A\overline{F}}_{A}} g[A]\right) \; = 0 \qquad (A \lessdot F),$$

$$\sum_{i=1}^{n} \left(g[G_1] \xrightarrow{\beta_{G_i,G_1}} g[G_i] \xrightarrow{\nu^{G_i}_{F_i}} g[F_i] \xrightarrow{\nu^{F_i}_{A}} g[A]\right) \; = 0 \qquad (\mathrm{rk}(X/A) = 2).$$