Algebraic Structure on Species

(A Categorical Approach)

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Species and Symmetric Sequences

[Joyal]

$$\mathrm{Set}^{\mathbb{B}} \simeq \mathrm{Set}^{\mathfrak{S}}$$

$\mathbb{B}$ = finite sets and bijections

$\mathfrak{S} = \coprod_{n \in \mathbb{N}} \mathfrak{S}_n$

Important: $\mathbb{B}, \mathfrak{S}$ freely generated by an object with symmetric monoidal structure

Constructions on Species

[Joyal]

Sums

$$\left(\coprod_{i\in I} X_i\right)(S) = \coprod_{i\in I} X_i(S)$$

Products

$$\left(\prod_{i\in I} X_i\right)(S) = \prod_{i\in I} X_i(S)$$

Representables

$$R : \mathbb{B} \hookrightarrow Set^{\mathbb{B}}$$

$$R(n) \overset{def}{=} \mathbb{B}(n,-)$$

Yoneda

$$\frac{R(n) \to X}{X(n)}$$

Multiplication = Day Convolution

$$\begin{tikzcd} \mathbb{B}\times\mathbb{B} \arrow[r, "R\times R"] \arrow[d, "+"'] \arrow[dr, phantom, "\cong"] & \mathrm{Set}^{\mathbb{B}}\times\mathrm{Set}^{\mathbb{B}} \arrow[d, "\bullet"] \\ \mathbb{B} \arrow[r, "R"'] & \mathrm{Set}^{\mathbb{B}} \end{tikzcd}$$

universal separatedly cocontinuous extension

$$R(m)\bullet R(n) = R(m+n) \qquad I = R0 = \text{"}1\text{"}$$

Formal construction (in the calculus of coends)

$$(X\bullet Y)(n) = \int^{n_1,n_2\in\mathbb{B}} X(n_1)\times Y(n_2)\times\mathbb{B}(n_1+n_2,n)$$

$$(x[\sigma_1], y[\sigma_2])\otimes\sigma \approx (x,y)\otimes(\sigma_1+\sigma_2)\sigma$$

Day Convolution generates closed structure

$$(-)\bullet X \dashv X \multimap (-)$$

$R1 = \text{"}x\text{"}$

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "R"] \arrow[d, "+1"'] & \mathrm{Set}^{\mathbb{B}} \arrow[d, "(-)\bullet R1"'] \arrow[d, phantom, shift left=3ex, "\dashv"] \\
\mathbb{B} \arrow[r, "R"'] & \mathrm{Set}^{\mathbb{B}} \arrow[u, bend right=60]
\end{tikzcd}$$

Derivation

$$R1 \multimap (-) = \partial$$

[Joyal]

$$(\partial X)(n) = X(n+1)$$

Important: Derivation has left and right adjoints

$$(-)\bullet R1 \dashv R1 \multimap (-) = \partial \dashv S$$

$$\begin{tikzcd}
\mathbb{A}^{\circ} \arrow[r, hook] \arrow[d] & \mathrm{Set}^{\mathbb{A}} \arrow[d, bend right=60] \arrow[d, phantom, shift right=3ex, "\dashv"] \arrow[d, phantom, shift left=3ex, "\dashv"] \arrow[d, bend left=60] \\
\mathbb{B}^{\circ} \arrow[r, hook] & \mathrm{Set}^{\mathbb{B}} \arrow[u]
\end{tikzcd}$$

Leibniz Rule

[Joyal]

$$\partial(X \cdot Y) \cong \partial(X) \cdot Y + X \cdot \partial(Y)$$

The canonical map

$$(R1 \multimap X) \cdot Y + X \cdot (R1 \multimap Y) \longrightarrow R1 \multimap (X \cdot Y)$$

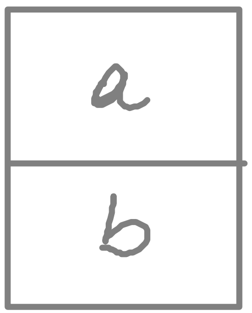
is invertible

$$\partial(X \cdot Y)(n) \cong \int^{n_1, n_2 \in \mathbb{B}} X(n_1) \times Y(n_2) \times \mathbb{B}(n_1 + n_2, n+1)$$

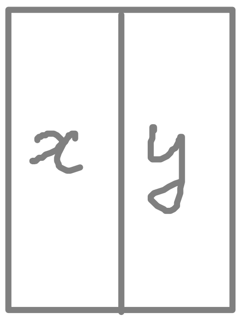
encapsulates the combinatorics of Leibniz Rule

Combinatorial Riesz Refinement Property

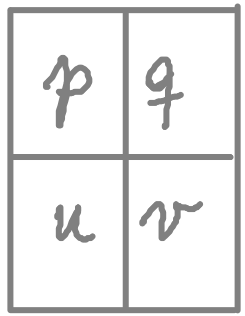
$$\mathbb{B}(a+b,\, x+y) \cong \int^{p,q,u,v \in \mathbb{B}} \mathbb{B}(a, p+q) \times \mathbb{B}(b, u+v) \times \mathbb{B}(p+u, x) \times \mathbb{B}(q+v, y)$$



$\cong$



iff $\exists$



$p+q \cong a \qquad u+v \cong b$

$p+u \cong x \qquad q+v \cong y$

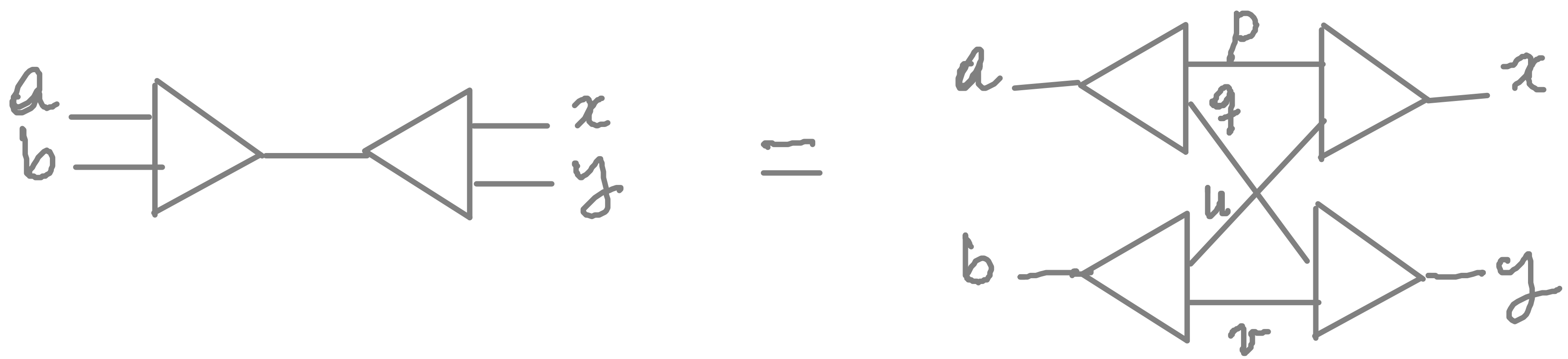
Combinatorial Riesz Refinement Property

$$\mathbb{B}(a+b, x+y)$$

$$\cong \int^{p,q,u,v \in \mathbb{B}} \mathbb{B}(a, p+q) \times \mathbb{B}(b, u+v) \times \mathbb{B}(p+u, x) \times \mathbb{B}(q+v, y)$$

Pictorially

Bialgebra Law



Combinatorial Riesz Refinement Property

$$\mathbb{B}(a+b, x+y)$$

$$\cong \int^{p,q,u,v \in \mathbb{B}} \mathbb{B}(a, p+q) \times \mathbb{B}(b, u+v) \times \mathbb{B}(p+u, x) \times \mathbb{B}(q+v, y)$$

$$\mathbb{B}(a+b, 1) \cong \mathbb{B}(a,1) \times \mathbb{B}(b,0) + \mathbb{B}(a,0) \times \mathbb{B}(b,1)$$

Combinatorial Equations

$$X \cong F(X)$$

combinatorial construction

Example: $F(X) = R1 + X \cdot X$

describe algebraic structure

$$F \circlearrowright \underline{\underline{Set}}^{\mathbb{B}}$$

algebra

$$\underset{\text{carrier}}{A \in \underline{\underline{Set}}^{\mathbb{B}}}, \quad \underset{\text{structure map}}{F(A) \to A} \text{ in } \underline{\underline{Set}}^{\mathbb{B}}$$

Example: $F(X) = R1 + X \cdot X$

$R1 + A \cdot A \to A$ equivalently

$$\begin{tikzcd}
R1 \arrow[dr, "\epsilon"'] & & A \cdot A \arrow[dl, "\omega"] \\
& A &
\end{tikzcd}$$

operations

$$\frac{}{x \vdash \epsilon_x} \qquad \frac{x_1 \dots x_m \vdash s \qquad y_1 \dots y_n \vdash t}{x_1 \dots x_m, y_1, \dots, y_n \vdash \omega(s,t)}$$

Homomorphisms

F-Alg

$$\begin{tikzcd} FA \arrow[r, "Fh"] \arrow[d, "\alpha"'] & FB \arrow[d, "\beta"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

Lambek's Lemma

Initial algebras have invertible structure maps.

Notation:

carrier = μF or $\mu X. F(X)$

structure = $F(\mu F) \xrightarrow[\cong]{\varphi} \mu F$

Structural Recursion

initiality

$$
\begin{tikzcd}
F(\mu F) \arrow[r, "F(\underline{it}\,\alpha)"] \arrow[d, "\varphi"', "\cong"] & FA \arrow[d, "\alpha"] \\
\mu F \arrow[r, dashed, "\underline{it}(\alpha)", "\exists!"'] & A
\end{tikzcd}
\quad \forall
$$

The iterator $\underline{it}(\alpha)$ is the unique map satisfying the fixpoint equation:

$$\underline{it}(\alpha) = \alpha \circ F(\underline{it}\,\alpha) \circ \varphi^{-1}$$

Iterative Construction

NB: If F preserves colimits of ω-chains, initial F-algebras may be calculated iteratively

$$
\begin{tikzcd}
0 \arrow[r] \arrow[drr] & F0 \arrow[r] \arrow[dr] & F^2 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & F^n 0 \arrow[r] \arrow[dll] & \cdots \\
& & \mu F & & &
\end{tikzcd}
$$

colimit

Free Algebras

$$
\begin{tikzcd}
F\text{-}Alg \arrow[d, shift left] \\
\underline{\underline{Set}}^B \arrow[u, bend left] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

free F-algebra on $A = \mu(A + F(-))$

Combinatorial Constructions

$$F(X) ::= X$$

$$\quad \mid \coprod_i F_i(X)$$

$$\frac{\coprod_i F_i(A) \xrightarrow{\alpha} A}{\left\{ F_i(A) \xrightarrow[\alpha_i]{} A \right\}}$$

Combinatorial Constructions

$$F(X) ::= X$$

$$\mid \coprod_i F_i(X)$$

$$\mid F_1(X) \cdot \cdots \cdot F_n(X)$$

$$F(A) \cdot G(A) \xrightarrow{\omega} A$$

$$\frac{m \vdash_F f \qquad n \vdash_G g}{m+n \vdash \omega(f,g)}$$

Combinatorial Constructions

$$F(X) ::= X$$
$$\mid \coprod_i F_i(X)$$
$$\mid F_1(X) \cdot \cdots \cdot F_n(X)$$
$$\mid F_1(X) \times \cdots \times F_n(X)$$

$$F(A) \times G(A) \xrightarrow{\omega} A$$

$$\frac{k \vdash_F f \qquad k \vdash_G g}{k \vdash \omega(f, g)}$$

Combinatorial Constructions

$$F(X) ::= X$$
$$\mid \coprod_i F_i(X)$$
$$\mid F_1(X) \cdot \cdots \cdot F_n(X)$$
$$\mid F_1(X) \times \cdots \times F_n(X)$$
$$\mid \partial(FX)$$

$$\partial(FA) \xrightarrow{\omega} A$$

Example:

$$\frac{x_1, \ldots, x_n, y \vdash \phi(x_1, \ldots, x_n, y)}{x_1, \ldots, x_n \vdash \forall y . \phi(x_1, \ldots, x_n, y)}$$

$$\frac{n+1 \vdash_F f}{n \vdash \omega(f)}$$

Importantly:

$$\begin{tikzcd}
F(A)(n+1) \arrow[r, "\omega_n"] \arrow[d, "F(A)(\pi+\mathrm{id})"] & A(n) \arrow[d, "A(\pi)"] \\
F(A)(m+1) \arrow[r, "\omega_m"'] & A(m)
\end{tikzcd}
\qquad
\begin{tikzcd}
n \arrow[d, "\pi", "\cong"'] \\
m
\end{tikzcd}$$

$$\omega_m(f[\pi+\mathrm{id}]) = (\omega_n(f))[\pi]$$

$$\partial(FA) \xrightarrow{\omega} A$$

Example:

$$\frac{x_1,\dots,x_n,y \vdash \phi(x_1,\dots,x_n,y)}{x_1,\dots,x_n \vdash \forall y.\,\phi(x_1,\dots,x_n,y)} \qquad \frac{n+1 \vdash_F f}{n \vdash \omega(f)}$$

Paradigmatic Example

Linear λ-calculus

[Tanaka]

$$F(X) = X \cdot X + \partial(X)$$

$$\frac{m \vdash s \qquad n \vdash t}{m+n \vdash s * t}$$

$$F(A) \longrightarrow A \quad \equiv \quad
\begin{tikzcd}
A \cdot A \arrow[dr, "*"] & \\
& A \\
\partial(A) \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

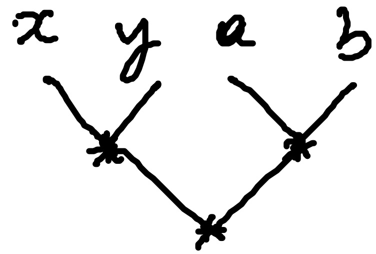
$$\frac{n+1 \vdash t}{n \vdash \lambda(t)}$$

Free F-algebra on $R1 = T$ $= \mu(R1 + F(-))$

$$\begin{tikzcd}
T \cdot T \arrow[dr, "*"] & & \\
& T & R1 \arrow[l, "v"'] \\
\partial(T) \arrow[ur, "\lambda"'] & &
\end{tikzcd}$$

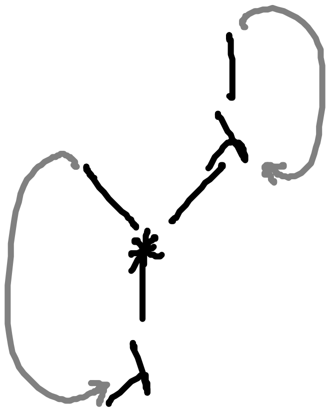
Example:

$$\dfrac{\dfrac{\overline{x \vdash v_x} \qquad \overline{y \vdash v_y}}{x, y \vdash v_x * v_y} \qquad \dfrac{\overline{a \vdash v_a} \qquad \overline{b \vdash v_b}}{a, b \vdash v_a * v_b}}{x, y, a, b \vdash (v_x * v_y) * (v_a * v_b)}$$



Examples:

$$\dfrac{\dfrac{\dfrac{}{x \vdash v_x} \qquad \dfrac{}{y \vdash v_y}}{x, y \vdash v_x * v_y} \qquad \dfrac{\dfrac{}{a \vdash v_a} \qquad \dfrac{}{b \vdash v_b}}{a, b \vdash v_a * v_b}}{x, y, a, b \vdash (v_x * v_y) * (v_a * v_b)}$$



$$\dfrac{\dfrac{\dfrac{}{x \vdash v_x} \qquad \dfrac{\dfrac{}{y \vdash v_y}}{\vdash \lambda_y(v_y)}}{x \vdash v_x * \lambda_y(v_y)}}{\vdash \lambda_x(v_x * (\lambda_y(v_y)))}$$

Substitution / Partial Composition / Grafting

$$\sigma : \partial(T) \cdot T \to T \quad , \quad T = \mu(R1 + F(-))$$

$$m+1 \vdash t,\; n \vdash s \;\mapsto\; m+n \vdash \sigma(t,s)$$

Examples

- Binary trees : $F(X) = X \cdot X$
- Linear λ-Terms : $F(X) = X \cdot X + \partial(X)$.

Recursive Definition?

$$\varphi = [\epsilon, *] : R1 + T\cdot T \xrightarrow{\cong} T$$

$$
\begin{tikzcd}[column sep=huge]
\partial(T)\cdot T \arrow[r, "\partial(\varphi^{-1})\cdot id", "\cong"'] \arrow[dd, "\sigma"'] \arrow[ddr, phantom, "?"] & \partial(R1 + T\cdot T)\cdot T \\
& \cong (R0 + \partial(T)\cdot T + T\cdot\partial(T))\cdot T \\
& \cong R0\cdot T + \partial(T)\cdot T\cdot T + T\cdot\partial(T)\cdot T \arrow[d, "{[\cong,\ \sigma\cdot id,\ id\cdot\sigma]}"] \\
T & T + T\cdot T + T\cdot T \arrow[l, "{[id,\ *,\ *]}"]
\end{tikzcd}
$$

Change perspective ...

$$
\begin{tikzcd}
\partial(T)\cdot T \arrow[r, shift left, "\partial(\varphi^{-1})\cdot id", "\cong"'] \arrow[ddd, "\sigma"'] & \partial(R1+T\cdot T)\cdot T \arrow[l, shift left, "\partial(\varphi)\cdot id"] \arrow[dd, bend right] \\
& \cong (R0+\partial(T)\cdot T + T\cdot\partial(T))\cdot T \\
& \cong R0\cdot T + \partial(T)\cdot T\cdot T + T\cdot\partial(T)\cdot T \arrow[d, "{[\cong,\ \sigma\cdot id,\ id\cdot\sigma]}"] \\
T & T + T\cdot T + T\cdot T \arrow[l, "{[id,\ *,\ *]}"]
\end{tikzcd}
$$

... look for an initiality property of $\partial(\varphi)$.

It is available because ∂ is a left adjoint!

Generalised Recursion [Bird & Paterson]

For a left adjoint L and a natural transformation

$$LG \overset{\ell}{\Longrightarrow} HL$$

if $\gamma : G(\mu G) \to G$ is an initial G-algebra then

$$\begin{tikzcd}
LG(\mu G) \arrow[r, "\ell"] \arrow[d, "L\gamma"', "\cong"] & HL(\mu G) \arrow[r, "H(\underline{git}\,\alpha)"] & HA \arrow[d, "\forall \alpha"] \\
L(\mu G) \arrow[rr, dashed, "\exists!\ \underline{git}(\alpha)"'] & & A
\end{tikzcd}$$

$$\underline{git}(\alpha) = \alpha \circ H(\underline{git}\,\alpha) \circ \ell \circ L(\gamma^{-1})$$

Thm: For $T = \mu(R1 + F(-))$ the free F-algebra on $R1$, $\partial(T)$ is an initial algebra.

Idea: For $F(X) = X \cdot X + \partial(X)$,

$(T, \partial(T))$ is the initial solution to

$$\begin{cases} X = R1 + X \cdot X + \partial(X) \\ Y = R0 + Y \cdot X + X \cdot Y + \partial(Y) \end{cases}$$

and

$\partial(T)$ is an initial $\mu(R0 + (-) \cdot T + T \cdot () + \partial(-))$

Substitution / Partial Composition is Algebraic Structure

<u>Def</u>: A linear substitution algebra is a structure

$$\begin{tikzcd} & R1 \arrow[dr, "\nu"] & & \partial(X)\cdot X \arrow[dl, "\sigma"'] \\ R0 \arrow[dr, "\underline{\nu}"'] & & X & \\ & \partial(X) & & \end{tikzcd}$$

satisfying

neutral element laws

$$\begin{tikzcd} R0\cdot X \arrow[d, "\underline{\nu}\cdot id"'] \arrow[dr, bend left, "\cong"] & \\ \partial(X)\cdot X \arrow[r, "\sigma"'] & X \end{tikzcd} \qquad \begin{tikzcd} \partial(X)\cdot R1 \cong (R1 \multimap X)\cdot R1 \arrow[d, "id\cdot\nu"'] \arrow[dr, "ev"] & \\ \partial(X)\cdot X \arrow[r, "\sigma"'] & X \end{tikzcd}$$

Associativity

$\sigma(t, \sigma(u,v)) = \sigma(\sigma(t,u), v)$

$$\begin{tikzcd}
\partial(X)\cdot\partial(X)\cdot X \arrow[rr] \arrow[d, "\mathrm{id}\cdot\sigma"'] & & \partial(\partial(X)\cdot X)\cdot X \arrow[d, "\partial(\sigma)\cdot\mathrm{id}"] \\
\partial(X)\cdot X \arrow[r, "\sigma"'] & X & \partial(X)\cdot X \arrow[l, "\sigma"]
\end{tikzcd}$$

Exchange

$\sigma(\sigma(t,u), v) = \sigma(\sigma(\underline{\mathrm{swap}}\, t, v), u)$

$$\begin{tikzcd}
\partial(\partial X)\cdot X\cdot X \arrow[rr] \arrow[d] & & \partial\partial(X)\cdot X\cdot X \arrow[d] \\
\partial(\partial(X)\cdot X)\cdot X \arrow[d, "\partial(\sigma)\cdot\mathrm{id}"'] & & \partial(\partial(X)\cdot X)\cdot X \arrow[d, "\partial(\sigma)\cdot\mathrm{id}"] \\
\partial(X)\cdot X \arrow[r, "\sigma"'] & X & \partial(X)\cdot X \arrow[l, "\sigma"]
\end{tikzcd}$$

Thm: Linear substitution algebras are, equivalently, symmetric operads

Thm: Free constructions

$$\begin{tikzcd} \mathrm{LinSubstAlg} \arrow[d, shift left=2] \\ \underline{\underline{\mathrm{Set}}}^{\mathbb{B}} \arrow[u, bend left=60] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

Linear Substitution Algebras with Algebraic Structure

Def: A linear substitution algebra with F-algebraic structure consists of

$$\text{lin. subst. alg} \qquad \begin{tikzcd} \partial(X)\cdot X \arrow[dr, "\sigma"'] & & FX \arrow[dl, "\chi"] \\ RI \arrow[r, "\nu"'] & X & \end{tikzcd} \qquad F\text{-alg.}$$

satisfying a compatibility condition

$$\begin{tikzcd} \partial(FX)\cdot X \arrow[r] \arrow[d, "\partial(\varphi)\cdot id"'] & \tilde{F}(\partial(X)\cdot X) \arrow[r, "\tilde{F}(\sigma)"] & \tilde{F}(X) \arrow[d, "\tilde{\varphi}"] \\ \partial(X)\cdot X \arrow[rr, "\sigma"'] & & X \end{tikzcd}$$

Example for a binary operation

$$
\begin{tikzcd}
\partial(X\cdot X)\cdot X \arrow[r] \arrow[ddd, "\partial(*)\cdot id"'] & (\partial(X)\cdot X + X\cdot\partial(X))\cdot X \arrow[d] & \\
& \partial(X)\cdot X\cdot X + X\cdot\partial(X)\cdot X \arrow[r, "{[\sigma\cdot id,\ id\cdot\sigma]}"'] & X\cdot X + X\cdot X \arrow[dd, "{[*,*]}"] \\
& & \\
\partial(X)\cdot X \arrow[rr, "\sigma"'] & & X
\end{tikzcd}
$$

$$
\sigma(m(t,s),u) = \begin{cases} m(\sigma(t,u),s) & \text{if pointing } t \\ m(t,\sigma(s,u)) & \text{if pointing } s \end{cases}
$$

Thm: For $F(X) = \underbrace{X \cdot \ldots \cdot X}_{n \text{ times}}$, a linear substitution algebra with F-algebraic structure is equivalently a symmetric operad with an n-ary operation.

- Given a binary operation $$\frac{\overline{\overline{R\,2 \longrightarrow X}}}{R\,0 \longrightarrow \partial\partial X}$$

we have

$$X \cdot X \cong R0 \cdot X \cdot X \longrightarrow \partial^2(X) \cdot X \cdot X \longrightarrow X$$

- Given $X \cdot X \longrightarrow X$ we have

$$R(2) \cong R(1) \cdot R(1) \longrightarrow X \cdot X \longrightarrow X$$

Thm: Free constructions [Fiore & Ranchod]

$$
\begin{tikzcd}
\mathbf{F\text{-}Lin\,Subst\,Alg} \arrow[d, shift left=2, "\mathbb{B}"] \\
\underline{\underline{\mathbf{Set}}} \arrow[u, bend left, shift left=2] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

In particular, the free F-algebra on R1,

$T = \mu(R1 + F(-))$ is initial in F-Lin Subst Alg

with substitution structure $\partial(T) \cdot T \longrightarrow T$

given by generalized structural recursion

as outlined.

Brief remarks on equational structure

- There is a general categorical theory of free constructions with equations.
  [Fiore & Hur]

- $X \bullet X \longrightarrow X \longleftarrow R\,0$ vs $X \times X \longrightarrow X \longleftarrow 1$

  monoid $\wr$ a monoid in species $\underline{\mathrm{Mon}}_{\bullet}(\underline{\underline{\mathrm{Set}}}^{\mathbb{B}})$

  monoid $\wr$ a species of monoids $\underline{\underline{\mathrm{Mon}}}^{\mathbb{B}}$

The pre-Lie Product

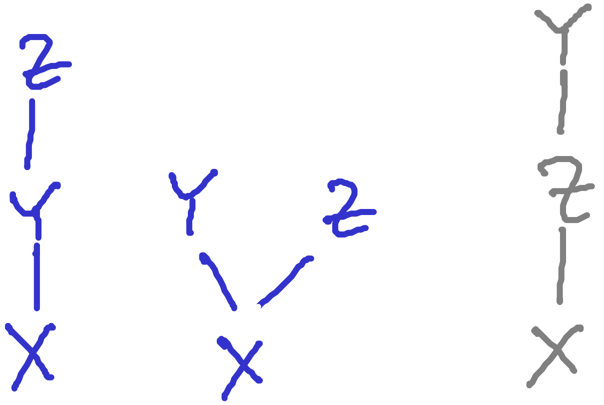
Def: The **pre-Lie product** of species is

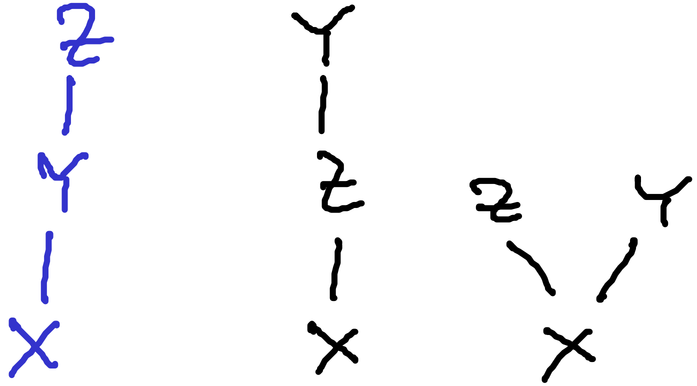
$$X * Y \overset{\text{def}}{=} \partial(X) \cdot Y$$

Pictorially:

- $(X * Y) * Z$ vs $X * (Y * Z)$?

$$(X * Y) * Z + X * (Z * Y) = X * (Y * Z) + (X * Z) * Y$$





Combinatorial Interpretation of the pre-Lie identity

Thm: There is a canonical natural pre-Lie isomorphism

$$(X*Y)*Z + X*(Z*Y) \overset{\pi}{\cong} X*(Y*Z) + (X*Z)*Y$$

Def: A π-algebra is a species A equipped with a structure map $A*A \xrightarrow{\alpha} A$ compatible with π.

Combinatorial Interpretation of the pre-Lie identity

Thm: There is a canonical natural pre-Lie isomorphism

$$(X*Y)*Z + X*(Z*Y) \overset{\pi}{\cong} X*(Y*Z) + (X*Z)*Y$$

Def: A π-algebra is a species A equipped with a structure map $A*A \xrightarrow{\alpha} A$ such that

$$\begin{tikzcd}
(A*A)*A + A*(A*A) \arrow[rr, "\overset{\pi}{\cong}"] \arrow[d, "{[\alpha*id,\, id*\alpha]}"'] & & A*(A*A) + (A*A)*A \arrow[d, "{[id*\alpha,\, \alpha*id]}"] \\
A*A \arrow[r, "\alpha"'] & A & A*A \arrow[l, "\alpha"]
\end{tikzcd}$$

Def: A Π-algebra is a species A equipped with a structure map $A * A \xrightarrow{\alpha} A$ s.t.

$$\begin{tikzcd}
(A*A)*A + A*(A*A) \arrow[r, "\overset{\pi}{\cong}"] \arrow[d, "{[\alpha * id,\, id * \alpha]}"'] & A*(A*A) + (A*A)*A \arrow[d, "{[id * \alpha,\, \alpha * id]}"] \\
A*A \arrow[r, "\alpha"'] & A \quad A*A \arrow[l, "\alpha"]
\end{tikzcd}$$

Thm: Π-algebras are equivalently non-unital symmetric operads.

Symmetric Matrices

Def: A symmetric matrix is an object of $\underline{\underline{\mathrm{Set}}}^{\mathbb{B}^{2}\times\mathbb{B}}$

Partial Derivatives

$$\partial_1(P)(m,n) =^{\mathrm{def}} P(m+1,n)$$

$$\partial_2(P)(m,n) =^{\mathrm{def}} P(m,n+1)$$

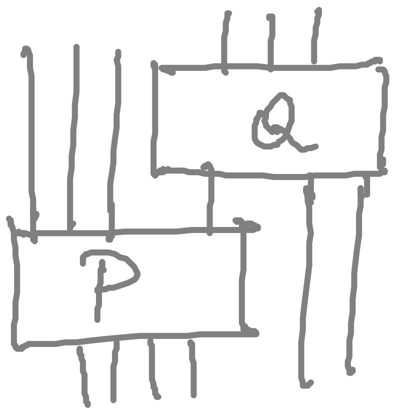
Lie-admissible Product

Def: $$P * Q \overset{def}{=} \partial_1(P) \cdot \partial_2(Q)$$

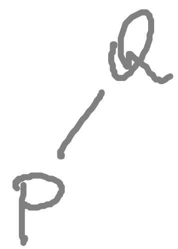
$$(P*Q)(m,n)$$

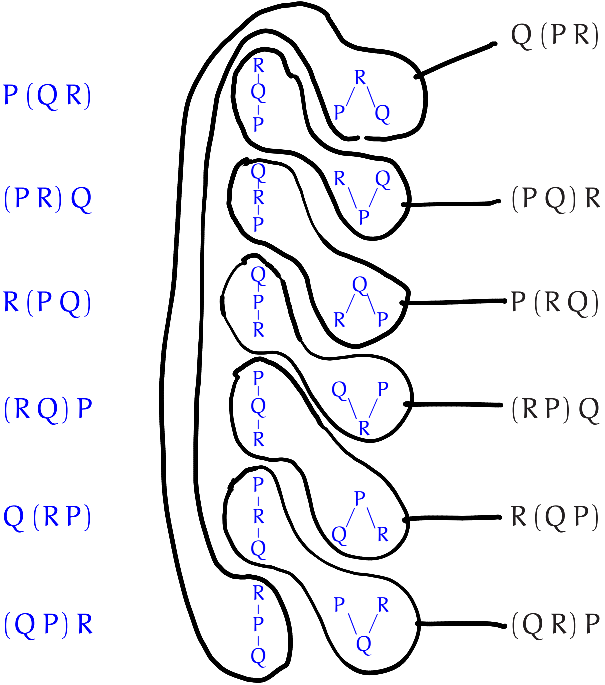
$$= \int^{m_1, m_2 \in \mathbb{B}^\circ,\ n_1, n_2 \in \mathbb{B}} P(m_1+1, n_1) \times Q(m_2, n_2+1) \times \mathbb{B}(m, m_1+m_2) \times \mathbb{B}(n_1+n_2, n)$$

Pictorially,



or




P (Q R)
(P R) Q
R (P Q)
(R Q) P
Q (R P)
(Q P) R
Q (P R)
(P Q) R
P (R Q)
(R P) Q
R (Q P)
(Q R) P

Combinatorial Interpretation of the Lie-admissible identity

Thm: There is a canonical Lie-admissible isomorphism

$$F(P,Q,R) \overset{\rho}{\cong} G(P,Q,R)$$

for

$$F(P,Q,R) = P*(Q*R) + (P*R)*Q + R*(P*Q) + (R*Q)*P + Q*(R*P) + (Q*P)*R$$

and

$$G(P,Q,R) = (P*Q)*R + Q*(P*R) + (Q*R)*P + R*(Q*P) + (R*P)*Q + P*(R*Q)$$

Def: A ρ-algebra is a symmetric matrix P equipped with a structure map $P * P \to P$ compatible with ρ; that is,

$$
\begin{tikzcd}
F(P,P,P) \arrow[r, phantom, "\overset{\rho}{\cong}"] \arrow[d] & G(P,P,P) \arrow[d] \\
P * P \arrow[r] & P & P * P \arrow[l]
\end{tikzcd}
$$

Thm: ρ-algebras are equivalently non-unital symmetric dioperads (one-object polycategories).